

# Extensions of the theory of tent spaces and applications to boundary value problems

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# Declaration

I hereby declare that the material in this thesis is my own original work except where stated otherwise.

Part **I** of this thesis consists of work that has either been published or has been submitted for publication. Chapter **1** has been published as ‘Tent spaces over metric measure spaces under doubling and related assumptions’ in the conference proceedings *Operator theory in harmonic and non-commutative analysis* [3]. Chapter **2** is joint work with Mikko Kemppainen, and has been published as ‘Non-uniformly local tent spaces’ in *Publicacions Matemàtiques* [5]. Chapter **3** has been submitted for publication under the title ‘Interpolation and embeddings of weighted tent spaces’, and is available as an arXiv preprint [4].

Part **II** will be submitted for publication, pending further additions and modifications.

A handwritten signature in black ink that reads "Alex Amenta". The script is fluid and cursive, with the first letters of the first and last names being capitalized and prominent.

Alex Amenta (Alexander Joseph Amenta)



# Acknowledgements

I would like to thank my supervisors, Pierre Portal (without whom this thesis would never have been started), and Pascal Auscher (without whom it would never have been finished). It has been an absolute pleasure to work with them, and I look forward to continuing our collaboration. I would also like to thank my honours supervisor, Bryan Wang, for (unintentionally) introducing me to research in analysis. I hope his influence is visible in this work, even 3.5 years after my time with him.

I realise as I write these acknowledgements that there is no way to thank all the friends who have helped me through this thesis—who truly deserve to be thanked individually—without forgetting anybody. Therefore, I thank them all not by name, but with the knowledge that they know who they are.

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...there is a sort of magic in the written word. The idea acquires substance by taking on a visible nature, and then stands in the way of its own clarification.

W. Somerset Maugham, *The Summing Up*

tout juste un peu de bruit  
pour combler le silence  
tout juste un peu de bruit  
et rien de plus

Brigitte Fontaine, *Comme à la radio*





## Abstract

We extend the theory of tent spaces from Euclidean spaces to various types of metric measure spaces. For doubling spaces we show that the usual ‘global’ theory remains valid, and for ‘non-uniformly locally doubling’ spaces (including  $\mathbb{R}^n$  with the Gaussian measure) we establish a satisfactory local theory. In the doubling context we show that Hardy–Littlewood–Sobolev-type embeddings hold in the scale of *weighted* tent spaces, and in the special case of unbounded AD-regular metric measure spaces we identify the real interpolants (the ‘ $Z$ -spaces’) of weighted tent spaces.

Weighted tent spaces and  $Z$ -spaces on  $\mathbb{R}^n$  are used to construct Hardy–Sobolev and Besov spaces adapted to perturbed Dirac operators. These spaces play a key role in the classification of solutions to first-order Cauchy–Riemann systems (or equivalently, the classification of conormal gradients of solutions to second-order elliptic systems) within weighted tent spaces and  $Z$ -spaces. We establish this classification, and as a corollary we obtain a useful characterisation of well-posedness of Regularity and Neumann problems for second-order complex-coefficient elliptic systems with boundary data in Hardy–Sobolev and Besov spaces of order  $s \in (-1, 0)$ .



# Contents

<b>Declaration</b>	<b>iii</b>
<b>Acknowledgements</b>	<b>v</b>
<b>Introduction</b>	<b>1</b>
<b>I Extensions of the theory of tent spaces</b>	<b>7</b>
<b>1 Tent spaces over doubling metric measure spaces</b>	<b>9</b>
1.1 Introduction . . . . .	9
1.2 Spatial assumptions . . . . .	10
1.3 The basic tent space theory . . . . .	12
1.3.1 Initial definitions and consequences . . . . .	12
1.3.2 Duality, the vector-valued approach, and complex interpolation . . . . .	20
1.3.3 Change of aperture . . . . .	32
1.3.4 Relations between $\mathcal{A}$ and $\mathcal{C}$ . . . . .	34
1.4 Appendix: Assorted lemmas and notation . . . . .	35
1.4.1 Tents, cones, and shadows . . . . .	35
1.4.2 Measurability . . . . .	38
1.4.3 Interpolation . . . . .	40
<b>2 Non-uniformly local tent spaces</b>	<b>41</b>
2.1 Introduction . . . . .	41
2.2 Weighted measures and admissible balls . . . . .	43
2.3 Local tent spaces: the reflexive range . . . . .	49
2.4 Endpoints: $\mathfrak{t}^{1,q}$ and $\mathfrak{t}^{\infty,q}$ . . . . .	51
2.4.1 Atomic decomposition . . . . .	52
2.4.2 Duality, interpolation and change of aperture . . . . .	55
2.5 Appendix 1: Local maximal functions . . . . .	57
2.6 Appendix 2: Cone covering lemma . . . . .	60
2.6.1 Review of non-negatively curved spaces . . . . .	60
2.6.2 Cone covering . . . . .	61

<b>3</b>	<b>Interpolation and embeddings of weighted tent spaces</b>	<b>65</b>
3.1	Introduction . . . . .	65
3.2	Preliminaries . . . . .	68
3.2.1	Metric measure spaces . . . . .	68
3.2.2	Unweighted tent spaces . . . . .	69
3.2.3	Weighted tent spaces: definitions, duality, and atoms . . . . .	73
3.3	Interpolation and embeddings . . . . .	77
3.3.1	Complex interpolation . . . . .	77
3.3.2	Real interpolation: the reflexive range . . . . .	80
3.3.3	Real interpolation: the non-reflexive range . . . . .	84
3.3.4	Hardy–Littlewood–Sobolev embeddings . . . . .	91
3.4	Deferred proofs . . . . .	94
3.4.1	$T^{p,\infty}$ – $L^\infty$ estimates for cylindrically supported functions . . . . .	94
3.4.2	$T^{p,\infty}$ atomic decomposition . . . . .	96
<b>II</b>	<b>Abstract Hardy–Sobolev and Besov spaces</b>	<b>101</b>
<b>4</b>	<b>Introduction</b>	<b>105</b>
4.1	Introduction and context . . . . .	105
4.1.1	Formulation of boundary value problems . . . . .	106
4.1.2	The first-order approach: perturbed Dirac operators and CR systems . . . . .	112
4.1.3	Solutions to CR systems and well-posedness . . . . .	118
4.2	Summary of the article . . . . .	121
4.3	Notation . . . . .	123
<b>5</b>	<b>Technical preliminaries</b>	<b>125</b>
5.1	Function space preliminaries . . . . .	125
5.1.1	Exponents . . . . .	125
5.1.2	Tent spaces . . . . .	127
5.1.3	$Z$ -spaces . . . . .	131
5.1.4	Unification: tent spaces, $Z$ -spaces, and slice spaces . . . . .	141
5.1.5	Homogeneous smoothness spaces . . . . .	148
5.2	Operator-theoretic preliminaries . . . . .	151
5.2.1	Bisectorial operators and holomorphic functional calculus . . . . .	151
5.2.2	Off-diagonal estimates and the Standard Assumptions . . . . .	156
5.2.3	Integral operators on tent spaces . . . . .	159
5.2.4	Extension and contraction operators . . . . .	164
<b>6</b>	<b>Adapted function spaces</b>	<b>171</b>
6.1	Adapted Hardy–Sobolev and Besov spaces . . . . .	171
6.1.1	Initial definitions, equivalent norms, and duality . . . . .	171
6.1.2	Mapping properties of holomorphic functional calculus . . . . .	176
6.1.3	Completions and interpolation . . . . .	177
6.1.4	The Cauchy operator on general adapted spaces . . . . .	182

6.2	Spaces adapted to perturbed Dirac operators . . . . .	185
6.2.1	Identification of spaces adapted to $D$ , $DB$ , and $BD$ . . . . .	186
6.2.2	The Cauchy operator on $DB$ -adapted spaces . . . . .	193
<b>7</b>	<b>Elliptic equations, CR systems, and boundary value problems</b>	<b>203</b>
7.1	Basic properties of solutions . . . . .	203
7.2	Decay of solutions at infinity . . . . .	205
7.3	Classification of solutions to Cauchy–Riemann systems . . . . .	207
7.3.1	Construction of solutions via Cauchy extension . . . . .	208
7.3.2	Initial limiting arguments . . . . .	209
7.3.3	Proof of Theorem 7.3.1 . . . . .	211
7.3.4	Proof of Theorem 7.3.2 . . . . .	216
7.4	Applications to boundary value problems . . . . .	230
7.4.1	Characterisation of well-posedness and corollaries . . . . .	230
7.4.2	The regularity problem for real coefficient scalar equations . . . . .	236
7.4.3	Additional boundary behaviour of solutions . . . . .	239
7.4.4	Layer potentials . . . . .	240



# Introduction

This thesis consists of two main parts. In the first part we provide various generalisations and extensions of the theory of tent spaces. In the second part we establish results concerning the well-posedness of certain elliptic boundary value problems, using some of our extended tent space theory in the process.

## Part I: Extensions of the theory of tent spaces

Tent spaces were first introduced by Coifman, Meyer, and Stein [32, 33] as a unification of fundamental ideas in modern harmonic analysis. Each of the three chapters of this part provides a different extension of their theory.

### Chapter 1: Tent spaces over metric measure spaces under doubling and related assumptions.

The main focus here is on doubling metric measure spaces  $(X, d, \mu)$ :  $(X, d)$  is a metric space,  $\mu$  is a Borel measure on  $(X, d)$ , and the doubling condition

$$\mu(B(x, 2r)) \lesssim \mu(B(x, r)) \quad (x \in X, r > 0)$$

is satisfied. We define tent spaces  $T^{p,q,\alpha}(X)$  associated with such a doubling metric measure space, and establish properties of  $T^{p,q,\alpha}(X)$  analogous to those established by Coifman, Meyer, and Stein in the case where  $X$  is  $\mathbb{R}^n$ ,  $d$  is the Euclidean distance, and  $\mu$  is the Lebesgue measure.

In particular, we show that these tent spaces are complete (Proposition 1.3.5), that the tent space scale is closed under duality (Propositions 1.3.10 and 1.3.15) and forms a complex interpolation scale (Propositions 1.3.12 and 1.3.18), and that the space  $T^{p,q,\alpha}(X)$  is independent of the ‘aperture’ parameter  $\alpha$  (Proposition 1.3.21). The proofs of these results are generally more technical than the corresponding Euclidean proofs, and we also point out that our proof of the complex interpolation result avoids an error in the original Coifman–Meyer–Stein argument.

The prototypical example of a doubling metric measure space is the Euclidean space  $\mathbb{R}^n$  with the Euclidean distance and Lebesgue measure. More generally, one can consider a Riemannian manifold of non-negative Ricci curvature, equipped with the geodesic distance and Riemannian volume (the curvature assumption ensures that the doubling condition is satisfied, by the Bishop–Gromov comparison theorem). Tent spaces associated with doubling Riemannian manifolds are the foundation for the Hardy spaces of differential forms developed by Auscher, McIntosh, and Russ [13] (see also the more recent work on this topic by Auscher, McIntosh, and Morris [11]). However, full details of this tent space theory had not appeared in the literature (with the exception of the atomic decomposition theorem, which was proven explicitly by Russ [81]). Therefore the material of this chapter fills a gap which was perhaps neglected in the past.

## Chapter 2: Non-uniformly local tent spaces.

In this chapter we consider metric measure spaces  $(X, d, \gamma)$  which are not doubling, but which are—in a certain quantified and non-uniform sense—locally doubling (for the precise definition see Section 2.2). Given such a space, we construct *non-uniformly local* tent spaces  $\mathfrak{t}_\alpha^{p,q}(\gamma)$ .<sup>1</sup> The main difference between these spaces and those constructed in Chapter 1 is that instead of the full ‘upper half-space’  $X \times \mathbb{R}_+$ , we use an *admissible region*  $D \subset X \times \mathbb{R}_+$  defined in terms of the ‘non-uniform local doubling’ data (see Definition 2.3.1).

Our theory of non-uniformly local tent spaces runs parallel to the theory constructed in Chapter 1. We also prove an atomic decomposition theorem (Theorem 2.4.5). Technicalities imposed on us by the non-uniform local doubling assumption force us to require that the metric space  $(X, d)$  is complete in the proof of this theorem.

The model non-uniformly locally doubling metric measure space is the Euclidean space  $\mathbb{R}^n$  equipped with the Euclidean distance and, in place of the Lebesgue measure, the *Gaussian measure*

$$d\gamma(x) = \frac{1}{(2\pi)^{n/2}} e^{-|x|^2/2} dx.$$

Non-uniformly local tent spaces associated with this space correspond to the *Gaussian tent spaces* defined by Maas, van Neerven, and Portal [63]. These are used in the construction of Gaussian Hardy spaces by Portal [78]. In Examples

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<sup>1</sup>Note that the notation has changed from the first article: such notation changes will occur in each article.



2.2.2 and 2.2.4 we provide many other examples of non-uniformly locally doubling spaces, given by weighted measures analogous to the Gaussian measure.

### Chapter 3: Interpolation and embeddings of weighted tent spaces.

Here we return to the setting of doubling metric measure spaces  $(X, d, \mu)$  as in Chapter 1. The tent space scale  $T^{p,q}(X)$  introduced there (we need not make reference to the aperture parameter  $\alpha$ , as we have already shown the tent spaces do not depend on it) is expanded: we define *weighted tent spaces*  $T_s^{p,q}(X)$  analogously to the spaces  $T^{p,q}(X) = T_0^{p,q}(X)$ , the difference being the presence of a weight  $\mu(B(x, t))^{-s}$  in the norm. This is motivated by applications to boundary value problems (which appear in Part II), where it is often natural to measure the function  $(t, x) \mapsto t^{-s} \nabla u(t, x)$  in  $T^{p,2}(\mathbb{R}^n)$  when  $u$  is the solution to an elliptic PDE.

The weighted tent space scale satisfies the following embedding property: when the parameters  $p_0, p_1, s_0, s_1$  satisfy the relation<sup>2</sup>

$$s_1 - s_0 = \frac{1}{p_1} - \frac{1}{p_0},$$

we have a continuous embedding

$$T_{s_0}^{p_0,q}(X) \hookrightarrow T_{s_1}^{p_1,q}(X)$$

(Theorem 3.3.19). These embeddings are actually quite counterintuitive. For homogeneous Sobolev spaces a similar embedding property (related to the Hardy–Littlewood–Sobolev lemma) holds, but this is interpreted as an interchange of regularity for integrability. In the context of weighted tent spaces, the parameter  $s$  does not actually reflect any kind of regularity.

When  $X$  is unbounded and AD-regular, so that in particular we have

$$\mu(B(x, r)) \simeq r^n \quad (x \in X, r > 0)$$

for some  $n > 0$ , we identify the real interpolation spaces

$$(T_{s_0}^{p_0,q}(X), T_{s_1}^{p_1,q}(X))_{\theta, p_\theta} = Z_{s_\theta}^{p_\theta,q}(X) \quad (1)$$

when  $p_0, p_1, q > 1$  (Theorem 3.3.4; see Definition 3.3.3 for the definition of the spaces  $Z_s^{p,q}(X)$ ). When  $X = \mathbb{R}^n$  we extend this result to  $p_0, p_1 > 0$  (Theorem

---

<sup>2</sup>Normally a factor of some ‘dimension’  $n$  should appear on the right hand side, but this does not appear here because of our convention of using ball volumes as weights.

3.3.9). The ‘Z-spaces’  $Z_s^{p,q}(X)$  are defined in terms of weighted  $L^p(X \times \mathbb{R}_+)$ -norms of  $L^q$  Whitney averages. They have appeared in the work of Barton and Mayboroda on elliptic boundary value problems with data in Besov spaces [21], but this connection with weighted tent spaces is new. Furthermore, this shows that Whitney averages arise naturally from the consideration of tent spaces, whereas in the past their use had always been justified by applications to PDE.

## Part II: Abstract Hardy–Sobolev and Besov spaces for elliptic boundary value problems with complex $L^\infty$ coefficients.

This part of the thesis, unlike the previous part, consists of one single (long) article. Broadly speaking, in this article we construct abstract Hardy–Sobolev and Besov spaces associated with perturbed Dirac operators, and we apply these spaces to the classification of solutions to Cauchy–Riemann systems. The foundation for our abstract Hardy–Sobolev and Besov spaces is the theory of weighted tent spaces (and their real interpolants, the Z-spaces) introduced in Chapter 3.

The main trajectory of this article follows the recent works of Auscher and Stahlhut [16] and Auscher and Mourgoglou [14]. However, we introduce many new techniques and shed some additional light on their results. For example, we introduce a new ‘exponent notation’, where boldface letters  $\mathbf{p}$  are used to denote pairs  $(p, s)$  or triples  $(\infty, s; \alpha)$ . The purpose of this notation is to combine integrability and regularity, and in turn to make the exponent calculations used in embeddings and interpolation more intuitive. We also refer to tent spaces  $T_s^p$  and Z-spaces  $Z_s^p$  simply as  $X^{\mathbf{p}}$ , in order to emphasise the fact that these spaces behave in essentially identical ways. This allows us to streamline our proofs, to handle spaces  $T_s^p$  and  $T_{s;\alpha}^\infty$  on an equal footing, and to prove results for Hardy–Sobolev and Besov spaces simultaneously.

A much more detailed overview of the article is contained in the introduction given there (Chapter 4).

## The structure of the thesis

As we have already pointed out, this thesis consists of four distinct articles, and each article uses different notational conventions. They may be read independently, although the later articles do refer to the earlier ones. Their bibliographies have been consolidated into one single bibliography. With the exception of

cosmetic changes and the correction of a few minor errors, the first two articles (Chapters [1](#) and [2](#)) are identical to the publications [\[3\]](#) and [\[5\]](#), and the third article (Chapter [3](#)) is identical to the preprint [\[4\]](#).



# Part I

## Extensions of the theory of tent spaces



# Chapter 1

## Tent spaces over metric measure spaces under doubling and related assumptions

### Abstract

In this article, we define the Coifman–Meyer–Stein tent spaces  $T^{p,q,\alpha}(X)$  associated with an arbitrary metric measure space  $(X, d, \mu)$  under minimal geometric assumptions. While gradually strengthening our geometric assumptions, we prove duality, interpolation, and change of aperture theorems for the tent spaces. Because of the inherent technicalities in dealing with abstract metric measure spaces, most proofs are presented in full detail.

### 1.1 Introduction

The purpose of this article is to indicate how the theory of tent spaces, as developed by Coifman, Meyer, and Stein for Euclidean space in [33], can be extended to more general metric measure spaces. Let  $X$  denote the metric measure space under consideration. If  $X$  is doubling, then the methods of [33] seem at first to carry over without much modification. However, there are some technicalities to be considered, even in this context. This is already apparent in the proof of the atomic decomposition given in [81].

Further still, there is an issue with the proof of the main interpolation result of [33] (see Remark 1.3.20 below). Alternate proofs of the interpolation result have since appeared in the literature — see for example [44], [23], [31], and [59]

— but these proofs are given in the Euclidean context, and no indication is given of their general applicability. In fact, the methods of [44] and [23] can be used to obtain a partial interpolation result under weaker assumptions than doubling. This result relies on some tent space duality; we show in Section 1.3.2 that this holds once we assume that the uncentred Hardy–Littlewood maximal operator is of strong type  $(r, r)$  for all  $r > 1$ .<sup>1</sup>

Finally, we consider the problem of proving the change of aperture result when  $X$  is doubling. The proof in [33] implicitly uses a geometric property of  $X$  which we term (NI), or ‘nice intersections’. This property is independent of doubling, but holds for many doubling spaces which appear in applications — in particular, all complete Riemannian manifolds have ‘nice intersections’. We provide a proof which does not require this assumption.

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## 1.2 Spatial assumptions

Throughout this article, we implicitly assume that  $(X, d, \mu)$  is a metric measure space; that is,  $(X, d)$  is a metric space and  $\mu$  is a Borel measure on  $X$ . The ball centred at  $x \in X$  of radius  $r > 0$  is the set

$$B(x, r) := \{y \in X : d(x, y) < r\},$$

and we write  $V(x, r) := \mu(B(x, r))$  for the volume of this set. We assume that the volume function  $V(x, r)$  is finite<sup>2</sup> and positive; one can show that  $V$  is automatically measurable on  $X \times \mathbb{R}_+$ .

There are four geometric assumptions which we isolate for future reference:

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<sup>1</sup>This fact is already implicit in [33].

<sup>2</sup>Since  $X$  is a metric space, this implies that  $\mu$  is  $\sigma$ -finite.



**(Proper)** a subset  $S \subset X$  is compact if and only if it is both closed and bounded, and the volume function  $V(x, r)$  is lower semicontinuous as a function of  $(x, r)$ ;<sup>3</sup>

**(HL)** the uncentred Hardy–Littlewood maximal operator  $\mathcal{M}$ , defined for measurable functions  $f$  on  $X$  by

$$\mathcal{M}(f)(x) := \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y)| d\mu(y) \quad (1.1)$$

where the supremum is taken over all balls  $B$  containing  $x$ , is of strong type  $(r, r)$  for all  $r > 1$ ;

**(Doubling)** there exists a constant  $C > 0$  such that for all  $x \in X$  and  $r > 0$ ,

$$V(x, 2r) \leq CV(x, r);$$

**(NI)** for all  $\alpha, \beta > 0$  there exists a positive constant  $c_{\alpha, \beta} > 0$  such that for all  $r > 0$  and for all  $x, y \in X$  with  $d(x, y) < \alpha r$ ,

$$\frac{\mu(B(x, \alpha r) \cap B(y, \beta r))}{V(x, \alpha r)} \geq c_{\alpha, \beta}.$$

We do not assume that  $X$  satisfies any of these assumptions unless mentioned otherwise. However, readers are advised to take  $(X, d, \mu)$  to be a complete Riemannian manifold with its geodesic distance and Riemannian volume if they are not interested in such technicalities.

It is well-known that doubling implies (HL). However, the converse is not true. See for example [37] and [82], where it is shown that (HL) is true for  $\mathbb{R}^2$  with the Gaussian measure. We will only consider (NI) along with doubling, so we remark that doubling does not imply (NI): one can see this by taking  $\mathbb{R}^2$  (now with Lebesgue measure) and removing an open strip.<sup>4</sup> One can show that all complete doubling length spaces—in particular, all complete doubling Riemannian manifolds—satisfy (NI).

---

<sup>3</sup>Note that this is a strengthening of the usual definition of a proper metric space, as the usual definition does not involve a measure. We have abused notation by using the word ‘proper’ in this way, as it is convenient in this context.

<sup>4</sup>One could instead remove an open bounded region with sufficiently regular boundary, for example an open square. This yields a connected example.

## 1.3 The basic tent space theory

### 1.3.1 Initial definitions and consequences

Let  $X^+$  denote the ‘upper half-space’  $X \times \mathbb{R}_+$ , equipped with the product measure  $d\mu(y) dt/t$  and the product topology. Since  $X$  and  $\mathbb{R}_+$  are metric spaces, with  $\mathbb{R}_+$  separable, the Borel  $\sigma$ -algebra on  $X^+$  is equal to the product of the Borel  $\sigma$ -algebras on  $X$  and  $\mathbb{R}_+$ , and so the product measure on  $X^+$  is Borel (see [26, Lemma 6.4.2(i)]).

We say that a subset  $C \subset X^+$  is *cylindrical* if it is contained in a cylinder: that is, if there exists  $x \in X$  and  $a, b, r > 0$  such that  $C \subset B(x, r) \times (a, b)$ . Note that cylindricity is equivalent to boundedness when  $X^+$  is equipped with an appropriate metric, and that compact subsets of  $X^+$  are cylindrical.

Cones and tents are defined as usual: for each  $x \in X$  and  $\alpha > 0$ , the *cone of aperture  $\alpha$  with vertex  $x$*  is the set

$$\Gamma^\alpha(x) := \{(y, t) \in X^+ : y \in B(x, \alpha t)\}.$$

For any subset  $F \subset X$  we write

$$\Gamma^\alpha(F) := \bigcup_{x \in F} \Gamma^\alpha(x).$$

For any subset  $O \subset X$ , the *tent of aperture  $\alpha$  over  $O$*  is defined to be the set

$$T^\alpha(O) := (\Gamma^\alpha(O^c))^c.$$

Writing

$$F_O(y, t) := \frac{\text{dist}(y, O^c)}{t} = t^{-1} \inf_{x \in O^c} d(y, x),$$

one can check that  $T^\alpha(O) = F_O^{-1}([\alpha, \infty))$ . Since  $F_O$  is continuous (due to the continuity of  $\text{dist}(\cdot, O^c)$ ), we find that tents are measurable, and so it follows that cones are also measurable.

Let  $F \subset X$  be such that  $O := F^c$  has finite measure. Given  $\gamma \in (0, 1)$ , we say that a point  $x \in X$  has *global  $\gamma$ -density with respect to  $F$*  if for all balls  $B$  containing  $x$ ,

$$\frac{\mu(B \cap F)}{\mu(B)} \geq \gamma.$$

We denote the set of all such points by  $F_\gamma^*$ , and define  $O_\gamma^* := (F_\gamma^*)^c$ . An important fact here is the equality

$$O_\gamma^* = \{x \in X : \mathcal{M}(\mathbf{1}_O)(x) > 1 - \gamma\},$$

where  $\mathbf{1}_O$  is the indicator function of  $O$ . We emphasise that  $\mathcal{M}$  denotes the *uncentred* maximal operator. When  $O$  is open (i.e. when  $F$  is closed), this shows that  $O \subset O_\gamma^*$  and hence that  $F_\gamma^* \subset F$ . Furthermore, the function  $\mathcal{M}(\mathbf{1}_O)$  is lower semicontinuous whenever  $\mathbf{1}_O$  is locally integrable (which is always true, since we assumed  $O$  has finite measure), which implies that  $F_\gamma^*$  is closed (hence measurable) and that  $O_\gamma^*$  is open (hence also measurable). Note that if  $X$  is doubling, then since  $\mathcal{M}$  is of weak-type  $(1, 1)$ , we have that

$$\mu(O_\gamma^*) \lesssim_{\gamma, X} \mu(O).$$

*Remark 1.3.1.* In our definition of points of  $\gamma$ -density, we used balls containing  $x$  rather than balls centred at  $x$  (as is usually done). This is done in order to avoid using the centred maximal function, which may not be measurable without assuming continuity of the volume function  $V(x, r)$ .

Here we find it convenient to introduce the notion of the  $\alpha$ -shadow of a subset of  $X^+$ . For a subset  $C \subset X^+$ , we define the  $\alpha$ -shadow of  $C$  to be the set

$$S^\alpha(C) := \{x \in X : \Gamma^\alpha(x) \cap C \neq \emptyset\}.$$

Shadows are always open, for if  $A \subset X^+$  is any subset, and if  $x \in S^\alpha(A)$ , then there exists a point  $(z, t_z) \in \Gamma^\alpha(x) \cap A$ , and one can easily show that  $B(x, \alpha t_z - d(x, z))$  is contained in  $S^\alpha(A)$ .

The starting point of the tent space theory is the definition of the operators  $\mathcal{A}_q^\alpha$  and  $\mathcal{C}_q^\alpha$ . For  $q \in (0, \infty)$ , the former is usually defined for measurable functions  $f$  on  $\mathbb{R}_+^{n+1}$  (with values in  $\mathbb{R}$  or  $\mathbb{C}$ , depending on context) by

$$\mathcal{A}_q^\alpha(f)(x)^q := \iint_{\Gamma^\alpha(x)} |f(y, t)|^q \frac{d\lambda(y) dt}{t^{n+1}}$$

where  $x \in \mathbb{R}^n$  and  $\lambda$  is Lebesgue measure. There are four reasonable ways to generalise this definition to our possibly non-doubling metric measure space  $X$ :<sup>5</sup> these take the form

$$\mathcal{A}_q^\alpha(f)(x)^q := \iint_{\Gamma^\alpha(x)} |f(y, t)|^q \frac{d\mu(y)}{V(\mathbf{a}, \mathbf{b}t)} \frac{dt}{t}$$

where  $\mathbf{a} \in \{x, y\}$  and  $\mathbf{b} \in \{1, \alpha\}$ . In all of these definitions, if a function  $f$  on  $X^+$  is supported on a subset  $C \subset X^+$ , then  $\mathcal{A}_q^\alpha(f)$  is supported on  $S^\alpha(C)$ ; we will use this fact repeatedly in what follows. Measurability of  $\mathcal{A}_q^\alpha(f)(x)$  in  $x$  when

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<sup>5</sup>We do not claim that these are the only reasonable generalisations.

$\mathbf{a} = y$  follows from Lemma 1.4.6 in the Appendix; the choice  $\mathbf{a} = x$  can be taken care of with a straightforward modification of this lemma. The choice  $\mathbf{a} = x$ ,  $\mathbf{b} = 1$  appears in [13, 81], and the choice  $\mathbf{a} = y$ ,  $\mathbf{b} = 1$  appears in [63, §3]. These definitions all lead to equivalent tent spaces when  $X$  is doubling. We will take  $\mathbf{a} = y$ ,  $\mathbf{b} = \alpha$  in our definition, as it leads to the following fundamental technique, which works with no geometric assumptions on  $X$ .

**Lemma 1.3.2** (Averaging trick). *Let  $\alpha > 0$ , and suppose  $\Phi$  is a nonnegative measurable function on  $X^+$ . Then*

$$\int_X \iint_{\Gamma^\alpha(x)} \Phi(y, t) \frac{d\mu(y)}{V(y, \alpha t)} \frac{dt}{t} d\mu(x) = \iint_{X^+} \Phi(y, t) d\mu(y) \frac{dt}{t}.$$

*Proof.* This is a straightforward application of Fubini–Tonelli’s theorem, which we present explicitly due to its importance in what follows:

$$\begin{aligned} & \int_X \iint_{\Gamma^\alpha(x)} \Phi(y, t) \frac{d\mu(y)}{V(y, \alpha t)} \frac{dt}{t} d\mu(x) \\ &= \int_X \int_0^\infty \int_X \mathbf{1}_{B(x, \alpha t)}(y) \Phi(y, t) \frac{d\mu(y)}{V(y, \alpha t)} \frac{dt}{t} d\mu(x) \\ &= \int_0^\infty \int_X \int_X \mathbf{1}_{B(y, \alpha t)}(x) d\mu(x) \Phi(y, t) \frac{d\mu(y)}{V(y, \alpha t)} \frac{dt}{t} \\ &= \int_0^\infty \int_X \frac{V(y, \alpha t)}{V(y, \alpha t)} \Phi(y, t) d\mu(y) \frac{dt}{t} \\ &= \iint_{X^+} \Phi(y, t) d\mu(y) \frac{dt}{t}. \end{aligned}$$

□

We will also need the following lemma in order to prove that our tent spaces are complete. Here we need to make some geometric assumptions.

**Lemma 1.3.3.** *Let  $X$  be proper or doubling. Let  $p, q, \alpha > 0$ , let  $K \subset X^+$  be cylindrical, and suppose  $f$  is a measurable function on  $X^+$ . Then*

$$\left\| \mathcal{A}_q^\alpha(\mathbf{1}_K f) \right\|_{L^p(X)} \lesssim \|f\|_{L^q(K)} \lesssim \left\| \mathcal{A}_q^\alpha(f) \right\|_{L^p(X)}, \quad (1.2)$$

with implicit constants depending on  $p, q, \alpha$ , and  $K$ .

*Proof.* Write

$$K \subset B(x, r) \times (a, b) =: C$$

for some  $x \in X$  and  $a, b, r > 0$ . We claim that there exist constants  $c_0, c_1 > 0$  such that for all  $(y, t) \in C$ ,

$$c_0 \leq V(y, \alpha t) \leq c_1.$$

If  $X$  is proper, this is an immediate consequence of the lower semicontinuity of the ball volume function (recall that we are assuming this whenever we assume  $X$  is proper) and the compactness of the closed cylinder  $\overline{B(x, r)} \times [a, b]$ . If  $X$  is doubling, then we argue as follows. Since  $V(y, \alpha t)$  is increasing in  $t$ , we have that

$$\min_{(y, t) \in C} V(y, \alpha t) \geq \min_{y \in B(x, r)} V(y, \alpha a)$$

and

$$\max_{(y, t) \in C} V(y, \alpha t) \leq \max_{y \in B(x, r)} V(y, \alpha b).$$

By the argument in the proof of Lemma 1.4.4 (in particular, by (1.16)), there exists  $c_0 > 0$  such that

$$\min_{y \in B(x, r)} V(y, \alpha a) \geq c_0.$$

Furthermore, since

$$V(y, \alpha b) \leq V(x, \alpha b + r)$$

for all  $y \in B(x, r)$ , we have that

$$\max_{y \in B(x, r)} V(y, \alpha b) \leq V(x, \alpha b + r) =: c_1,$$

proving the claim.

To prove the first estimate of (1.2), write

$$\begin{aligned} \left\| \mathcal{A}_q^\alpha(\mathbf{1}_K f) \right\|_{L^p(X)} &= \left( \int_{S^\alpha(K)} \left( \iint_{\Gamma^\alpha(x)} \mathbf{1}_K(y, t) |f(y, t)|^q \frac{d\mu(y)}{V(y, \alpha t)} \frac{dt}{t} \right)^{\frac{p}{q}} d\mu(x) \right)^{\frac{1}{p}} \\ &\lesssim_{c_0, q} \left( \int_{S^\alpha(K)} \left( \iint_K |f(y, t)|^q d\mu(y) \frac{dt}{t} \right)^{\frac{p}{q}} d\mu(x) \right)^{\frac{1}{p}} \\ &\lesssim_K \|f\|_{L^q(K)}. \end{aligned}$$

To prove the second estimate, first choose finitely many points  $(x_n)_{n=1}^N$  such that

$$\overline{B(x, r)} \subset \bigcup_{n=1}^N B(x_n, \alpha a/2)$$

using either compactness of  $\overline{B(x, r)}$  (in the proper case) or doubling.<sup>6</sup> Write  $B_n := B(x_n, \alpha a/2)$ . We then have

$$\begin{aligned} \left( \iint_K |f(y, t)|^q d\mu(y) \frac{dt}{t} \right)^{\frac{1}{q}} &\lesssim_{c_1} \left( \iint_K \sum_{n=1}^N \mathbf{1}_{B_n}(y) |f(y, t)|^q \frac{d\mu(y)}{V(y, \alpha t)} \frac{dt}{t} \right)^{\frac{1}{q}} \\ &\lesssim_{X, q} \sum_{n=1}^N \left( \iint_K \mathbf{1}_{B_n}(y) |f(y, t)|^q \frac{d\mu(y)}{V(y, \alpha t)} \frac{dt}{t} \right)^{\frac{1}{q}}. \end{aligned}$$

If  $x, y \in B_n$ , then  $d(x, y) < \alpha a < \alpha t$  (since  $t > a$ ), and so

$$\iint_K \mathbf{1}_{B_n}(y) |f(y, t)|^q \frac{d\mu(y)}{V(y, \alpha t)} \frac{dt}{t} \leq \iint_{\Gamma^\alpha(x)} |f(y, t)|^q \frac{d\mu(y)}{V(y, \alpha t)} \frac{dt}{t}. \quad (1.3)$$

We then have

$$\begin{aligned} &\sum_{n=1}^N \left( \iint_K \mathbf{1}_{B_n}(y) |f(y, t)|^q \frac{d\mu(y)}{V(y, \alpha t)} \frac{dt}{t} \right)^{1/q} \\ &= \sum_{n=1}^N \left( \int_{B_n} \left( \iint_K \mathbf{1}_{B_n}(y) |f(y, t)|^q \frac{d\mu(y)}{V(y, \alpha t)} \frac{dt}{t} \right)^{p/q} d\mu(x) \right)^{1/p} \\ &\leq \sum_{n=1}^N \left( \int_{B_n} \left( \iint_{\Gamma^\alpha(x)} |f(y, t)|^q \frac{d\mu(y)}{V(y, \alpha t)} \frac{dt}{t} \right)^{p/q} d\mu(x) \right)^{1/p} \\ &\leq N \left( \max_n \mu(B_n)^{-1/p} \right) \left( \int_X \left( \iint_{\Gamma^\alpha(x)} |f(y, t)|^q \frac{d\mu(y)}{V(y, \alpha t)} \frac{dt}{t} \right)^{p/q} d\mu(x) \right)^{1/p} \\ &\lesssim_{K, p, \alpha} \left\| \mathcal{A}_q^\alpha(f) \right\|_{L^p(X)}, \end{aligned}$$

which completes the proof.  $\square$

As usual, with  $\alpha > 0$  and  $p, q \in (0, \infty)$ , we define the tent space (quasi-)norm of a measurable function  $f$  on  $X^+$  by

$$\|f\|_{T^{p, q, \alpha}(X)} := \left\| \mathcal{A}_q^\alpha(f) \right\|_{L^p(X)},$$

and the tent space  $T^{p, q, \alpha}(X)$  to be the (quasi-)normed vector space consisting of all such  $f$  (defined almost everywhere) for which this quantity is finite.

*Remark 1.3.4.* One can define the tent space as either a real or complex vector space, according to one's own preference. We will implicitly work in the complex setting (so our functions will always be  $\mathbb{C}$ -valued). Apart from complex interpolation, which demands that we consider complex Banach spaces, the difference is immaterial.

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<sup>6</sup>In the doubling case, this is a consequence of what is usually called ‘geometric doubling’. A proof that this follows from the doubling condition can be found in [34, §III.1].

**Proposition 1.3.5.** *Let  $X$  be proper or doubling. For all  $p, q, \alpha \in (0, \infty)$ , the tent space  $T^{p,q,\alpha}(X)$  is complete and contains  $L_c^q(X^+)$  (the space of functions  $f \in L^q(X^+)$  with cylindrical support) as a dense subspace.*

*Proof.* Let  $(f_n)_{n \in \mathbb{N}}$  be a Cauchy sequence in  $T^{p,q,\alpha}(X)$ . Then by Lemma 1.3.3, for every cylindrical subset  $K \subset X^+$  the sequence  $(\mathbf{1}_K f_n)_{n \in \mathbb{N}}$  is Cauchy in  $L^q(K)$ . We thus obtain a limit

$$f_K := \lim_{n \rightarrow \infty} \mathbf{1}_K f_n \in L^q(K)$$

for each  $K$ . If  $K_1$  and  $K_2$  are two cylindrical subsets of  $X^+$ , then  $f_{K_1}|_{K_1 \cap K_2} = f_{K_2}|_{K_1 \cap K_2}$ , so by making use of an increasing sequence  $\{K_m\}_{m \in \mathbb{N}}$  of cylindrical subsets of  $X^+$  whose union is  $X^+$  (for example, we could take  $K_m := B(x, m) \times (1/m, m)$  for some  $x \in X$ ) we obtain a function  $f \in L_{\text{loc}}^q(X^+)$  with  $f|_{K_m} = f_{K_m}$  for each  $m \in \mathbb{N}$ .<sup>7</sup> This is our candidate limit for the sequence  $(f_n)_{n \in \mathbb{N}}$ .

To see that  $f$  lies in  $T^{p,q,\alpha}(X)$ , write for any  $m, n \in \mathbb{N}$

$$\begin{aligned} \|\mathbf{1}_{K_m} f\|_{T^{p,q,\alpha}(X)} &\lesssim_{p,q} \|\mathbf{1}_{K_m}(f - f_n)\|_{T^{p,q,\alpha}(X)} + \|\mathbf{1}_{K_m} f_n\|_{T^{p,q,\alpha}(X)} \\ &\leq C_{p,q,\alpha,X,m} \|f - f_n\|_{L^q(K_m)} + \|f_n\|_{T^{p,q,\alpha}(X)}, \end{aligned}$$

the  $(p, q)$ -dependence in the first estimate being relevant only for  $p < 1$  or  $q < 1$ , and the second estimate coming from Lemma 1.3.3. Since the sequence  $(f_n)_{n \in \mathbb{N}}$  converges to  $\mathbf{1}_{K_m} f$  in  $L^q(K_m)$  and is Cauchy in  $T^{p,q,\alpha}(X)$ , we have that

$$\|\mathbf{1}_{K_m} f\|_{T^{p,q,\alpha}(X)} \lesssim \sup_{n \in \mathbb{N}} \|f_n\|_{T^{p,q,\alpha}(X)}$$

uniformly in  $m$ . Hence  $\|f\|_{T^{p,q,\alpha}(X)}$  is finite.

We now claim that for all  $\varepsilon > 0$  there exists  $m \in \mathbb{N}$  such that for all sufficiently large  $n \in \mathbb{N}$ , we have

$$\|\mathbf{1}_{K_m^c}(f_n - f)\|_{T^{p,q,\alpha}(X)} \leq \varepsilon.$$

Indeed, since the sequence  $(f_n)_{n \in \mathbb{N}}$  is Cauchy in  $T^{p,q,\alpha}(X)$ , there exists  $N \in \mathbb{N}$  such that for all  $n, n' \geq N$  we have  $\|f_n - f_{n'}\|_{T^{p,q,\alpha}(X)} < \varepsilon/2$ . Furthermore, since

$$\lim_{m \rightarrow \infty} \|\mathbf{1}_{K_m^c}(f_N - f)\|_{T^{p,q,\alpha}(X)} = 0$$

by the Dominated Convergence Theorem, we can choose  $m$  such that

$$\|\mathbf{1}_{K_m^c}(f_N - f)\|_{T^{p,q,\alpha}(X)} < \varepsilon/2.$$

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<sup>7</sup>We interpret ‘locally integrable on  $X^+$ ’ as meaning ‘integrable on all cylinders’, rather than ‘integrable on all compact sets’.

Then for all  $n \geq N$ ,

$$\begin{aligned} \|\mathbf{1}_{K_m^c}(f_n - f)\|_{T^{p,q,\alpha}(X)} &\lesssim_{p,q} \|\mathbf{1}_{K_m^c}(f_n - f_N)\|_{T^{p,q,\alpha}(X)} + \|\mathbf{1}_{K_m^c}(f_N - f)\|_{T^{p,q,\alpha}(X)} \\ &\leq \|f_n - f_N\|_{T^{p,q,\alpha}(X)} + \|\mathbf{1}_{K_m^c}(f_N - f)\|_{T^{p,q,\alpha}(X)} \\ &< \varepsilon, \end{aligned}$$

proving the claim.

Finally, by the previous remark, for all  $\varepsilon > 0$  we can find  $m$  such that for all sufficiently large  $n \in \mathbb{N}$  we have

$$\begin{aligned} \|f_n - f\|_{T^{p,q,\alpha}(X)} &\lesssim_{p,q} \|\mathbf{1}_{K_m}(f_n - f)\|_{T^{p,q,\alpha}(X)} + \|\mathbf{1}_{K_m^c}(f_n - f)\|_{T^{p,q,\alpha}(X)} \\ &< \|\mathbf{1}_{K_m}(f_n - f)\|_{T^{p,q,\alpha}(X)} + \varepsilon \\ &\leq C(p, q, \alpha, X, m) \|f_n - f\|_{L^q(K_m)} + \varepsilon. \end{aligned}$$

Taking the limit of both sides as  $n \rightarrow \infty$ , we find that  $\lim_{n \rightarrow \infty} f_n = f$  in  $T^{p,q,\alpha}(X)$ , and therefore  $T^{p,q,\alpha}(X)$  is complete.

To see that  $L_c^q(X^+)$  is dense in  $T^{p,q,\alpha}(X)$ , simply write  $f \in T^{p,q,\alpha}(X)$  as the pointwise limit

$$f = \lim_{n \rightarrow \infty} \mathbf{1}_{K_n} f.$$

By the Dominated Convergence Theorem, this convergence holds in  $T^{p,q,\alpha}(X)$ .  $\square$

We note that Lemma 1.3.2 implies that in the case where  $p = q$ , we have  $T^{p,p,\alpha}(X) = L^p(X^+)$  for all  $\alpha > 0$ .

In the same way as Lemma 1.3.2, we can prove the analogue of [33, Lemma 1].

**Lemma 1.3.6** (First integration lemma). *For any nonnegative measurable function  $\Phi$  on  $X^+$ , with  $F$  a measurable subset of  $X$  and  $\alpha > 0$ ,*

$$\int_F \iint_{\Gamma^\alpha(x)} \Phi(y, t) d\mu(y) dt d\mu(x) \leq \iint_{\Gamma^\alpha(F)} \Phi(y, t) V(y, \alpha t) d\mu(y) dt.$$

*Remark 1.3.7.* There is one clear disadvantage of our choice of tent space norm: it is no longer clear that

$$\|\cdot\|_{T^{p,q,\alpha}(X)} \leq \|\cdot\|_{T^{p,q,\beta}(X)} \quad (1.4)$$

when  $\alpha < \beta$ . In fact, this may not even be true for general non-doubling spaces. This is no great loss, since for doubling spaces we can revert to the ‘original’ tent space norm (with  $\mathbf{a} = x$  and  $\mathbf{b} = 1$ ) at the cost of a constant depending only on  $X$ , and for this choice of norm (1.4) is immediate.



In order to define the tent spaces  $T^{\infty,q,\alpha}(X)$ , we need to introduce the operator  $\mathcal{C}_q^\alpha$ . For measurable functions  $f$  on  $X^+$ , we define

$$\mathcal{C}_q^\alpha(f)(x) := \sup_{B \ni x} \left( \frac{1}{\mu(B)} \iint_{T^\alpha(B)} |f(y,t)|^q d\mu(y) \frac{dt}{t} \right)^{\frac{1}{q}},$$

where the supremum is taken over all balls containing  $x$ . Since  $\mathcal{C}_q^\alpha(f)$  is lower semicontinuous (see Lemma 1.4.7),  $\mathcal{C}_q^\alpha(f)$  is measurable. For functions  $f$  on  $X^+$  we define the (quasi-)norm  $\|\cdot\|_{T^{\infty,q,\alpha}(X)}$  by

$$\|f\|_{T^{\infty,q,\alpha}(X)} := \left\| \mathcal{C}_q^\alpha(f) \right\|_{L^\infty(X)},$$

and the tent space  $T^{\infty,q,\alpha}(X)$  as the (quasi-)normed vector space of measurable functions  $f$  on  $X^+$ , defined almost everywhere, for which  $\|f\|_{T^{\infty,q,\alpha}(X)}$  is finite. The proof that  $T^{\infty,q,\alpha}(X)$  is a (quasi-)Banach space is similar to that of Proposition 1.3.5 once we have established the following analogue of Lemma 1.3.3.

**Lemma 1.3.8.** *Let  $q, \alpha > 0$ , let  $K \subset X^+$  be cylindrical, and suppose  $f$  is a measurable function on  $X^+$ . Then*

$$\|f\|_{L^q(K)} \lesssim \|f\|_{T^{\infty,q,\alpha}(X)}, \quad (1.5)$$

with implicit constant depending only on  $\alpha, q$ , and  $K$  (but not otherwise on  $X$ ).

Furthermore, if  $X$  is proper or doubling, then we also have

$$\|\mathbf{1}_K f\|_{T^{\infty,q,\alpha}(X)} \lesssim \|f\|_{L^q(K)},$$

again with implicit constant depending only on  $\alpha, q$ , and  $K$ .

*Proof.* We use Lemma 1.4.4. To prove the first estimate, for each  $\varepsilon > 0$  we can choose a ball  $B_\varepsilon$  such that  $T^\alpha(B_\varepsilon) \supset K$  and  $\mu(B_\varepsilon) < \beta_1(K) + \varepsilon$ . Then

$$\begin{aligned} \|f\|_{L^q(K)} &\leq \left\| \mathbf{1}_{T^\alpha(B_\varepsilon)} f \right\|_{L^q(X^+)} \\ &= \mu(B_\varepsilon)^{\frac{1}{q}} \mu(B_\varepsilon)^{-\frac{1}{q}} \left\| \mathbf{1}_{T^\alpha(B_\varepsilon)} f \right\|_{L^q(X^+)} \\ &\leq (\beta_1(K) + \varepsilon)^{\frac{1}{q}} \|f\|_{T^{\infty,q,\alpha}(X)}. \end{aligned}$$

In the final line we used that  $\mu(B_\varepsilon) > 0$  to conclude that

$$\mu(B_\varepsilon)^{-1/q} \left\| \mathbf{1}_{T^\alpha(B_\varepsilon)} f \right\|_{L^q(X^+)}$$

is less than the *essential* supremum of  $\mathcal{C}_q^\alpha(f)$ . Since  $\varepsilon > 0$  was arbitrary, we have the first estimate.

For the second estimate, assuming that  $X$  is proper or doubling, observe that

$$\begin{aligned} \|\mathbf{1}_K f\|_{T^{\infty,q,\alpha}(X)} &\leq \sup_{B \subset X} \left( \frac{1}{\mu(B)} \iint_{T^\alpha(B) \cap K} |f(y,t)|^q d\mu(y) \frac{dt}{t} \right)^{\frac{1}{q}} \\ &\leq \left( \frac{1}{\beta_0(K)} \iint_K |f(y,t)|^q d\mu(y) \frac{dt}{t} \right)^{\frac{1}{q}} \\ &= \beta_0(K)^{-\frac{1}{q}} \|f\|_{L^q(K)}, \end{aligned}$$

completing the proof.  $\square$

*Remark 1.3.9.* In this section we did not impose any geometric conditions on our space  $X$  besides our standing assumptions on the measure  $\mu$  and the properness assumption (in the absence of doubling). Thus we have defined the tent space  $T^{p,q,\alpha}(X)$  in considerable generality. However, what we have defined is a *global* tent space, and so this concept may not be inherently useful when  $X$  is non-doubling. Instead, our interest is to determine precisely where geometric assumptions are needed in the tent space theory.

### 1.3.2 Duality, the vector-valued approach, and complex interpolation

#### Midpoint results

The geometric assumption (HL) from Section 1.2 now comes into play. For  $r \geq 0$ , we denote the Hölder conjugate of  $r$  by  $r' := r/(r-1)$  with  $r' = \infty$  when  $r = 1$ .

**Proposition 1.3.10.** *Suppose that  $X$  is either proper or doubling, and satisfies assumption (HL). Then for  $p, q \in (1, \infty)$  and  $\alpha > 0$ , the pairing*

$$\langle f, g \rangle := \iint_{X^+} f(y,t) \overline{g(y,t)} d\mu(y) \frac{dt}{t} \quad (f \in T^{p,q,\alpha}(X), g \in T^{p',q',\alpha}(X))$$

*realises  $T^{p',q',\alpha}(X)$  as the Banach space dual of  $T^{p,q,\alpha}(X)$ , up to equivalence of norms.*

This is proved in the same way as in [33]. We provide the details in the interest of self-containment.

*Proof.* We first remark that if  $p = q$ , the duality statement is a trivial consequence of the equality  $T^{p,p,\alpha}(X) = L^p(X^+)$ .

In general, suppose  $f \in T^{p,q,\alpha}(X)$  and  $g \in T^{p',q',\alpha}(X)$ . Then by the averaging trick and Hölder's inequality, we have

$$\begin{aligned} |\langle f, g \rangle| &\leq \int_X \iint_{\Gamma^\alpha(x)} |f(y, t) \overline{g(y, t)}| \frac{d\mu(y)}{V(y, \alpha t)} \frac{dt}{t} d\mu(x) \\ &\leq \int_X \mathcal{A}_q^\alpha(f)(x) \mathcal{A}_{q'}^\alpha(g)(x) d\mu(x) \\ &\leq \|f\|_{T^{p,q,\alpha}(X)} \|g\|_{T^{p',q',\alpha}(X)}. \end{aligned} \tag{1.6}$$

Thus every  $g \in T^{p',q',\alpha}(X)$  induces a bounded linear functional on  $T^{p,q,\alpha}(X)$  via the pairing  $\langle \cdot, \cdot \rangle$ , and so  $T^{p',q',\alpha}(X) \subset (T^{p,q,\alpha}(X))^*$ .

Conversely, suppose  $\ell \in (T^{p,q,\alpha}(X))^*$ . If  $K \subset X^+$  is cylindrical, then by the properness or doubling assumption, we can invoke Lemma 1.3.3 to show that  $\ell$  induces a bounded linear functional  $\ell_K \in (L^q(K))^*$ , which can in turn be identified with a function  $g_K \in L^{q'}(K)$ . By covering  $X^+$  with an increasing sequence of cylindrical subsets, we thus obtain a function  $g \in L_{\text{loc}}^{q'}(X^+)$  such that  $g|_K = g_K$  for all cylindrical  $K \subset X^+$ .

If  $f \in L^q(X^+)$  is cylindrically supported, then we have

$$\begin{aligned} \iint_{X^+} f(y, t) \overline{g(y, t)} d\mu(y) \frac{dt}{t} &= \iint_{\text{supp } f} f(y, t) \overline{g_{\text{supp } f}(y, t)} d\mu(y) \frac{dt}{t} \\ &= \ell_{\text{supp } f}(f) \\ &= \ell(f), \end{aligned} \tag{1.7}$$

recalling that  $f \in T^{p,q,\alpha}(X)$  by Lemma 1.3.3. Since the cylindrically supported  $L^q(X^+)$  functions are dense in  $T^{p,q,\alpha}(X)$ , the representation (1.7) of  $\ell(f)$  in terms of  $g$  is valid for all  $f \in T^{p,q,\alpha}(X)$  by dominated convergence and the inequality (1.6), provided we show that  $g$  is in  $T^{p',q',\alpha}(X)$ .

Now suppose  $p < q$ . We will show that  $g$  lies in  $T^{p',q',\alpha}(X)$ , thus showing directly that  $(T^{p,q,\alpha}(X))^*$  is contained in  $T^{p',q',\alpha}(X)$ . It suffices to show this for  $g_K$ , where  $K \subset X^+$  is an arbitrary cylindrical subset, provided we obtain an estimate which is uniform in  $K$ . We estimate

$$\|g_K\|_{T^{p',q',\alpha}(X)}^{q'} = \left\| \mathcal{A}_{q'}^\alpha(g_K)^{q'} \right\|_{L^{p'/q'}(X)}$$

by duality. Let  $\psi \in L^{(p'/q')'}(X)$  be nonnegative, with  $\|\psi\|_{L^{(p'/q')'}(X)} \leq 1$ . Then by

Fubini–Tonelli’s theorem,

$$\begin{aligned}
& \int_X \mathcal{A}_{q'}^\alpha(g_K)(x)^{q'} \psi(x) d\mu(x) \\
&= \int_X \iint_{X^+} \mathbf{1}_{B(y, \alpha t)}(x) |g_K(y, t)|^{q'} \frac{d\mu(y)}{V(y, \alpha t)} \frac{dt}{t} \psi(x) d\mu(x) \\
&= \int_0^\infty \int_X \frac{1}{V(y, \alpha t)} \int_{B(y, \alpha t)} \psi(x) d\mu(x) |g_K(y, t)|^{q'} d\mu(y) \frac{dt}{t} \\
&= \iint_{X^+} M_{\alpha t} \psi(y) |g_K(y, t)|^{q'} d\mu(y) \frac{dt}{t},
\end{aligned}$$

where  $M_s$  is the averaging operator defined for  $y \in X$  and  $s > 0$  by

$$M_s \psi(y) := \frac{1}{V(y, s)} \int_{B(y, s)} \psi(x) d\mu(x).$$

Thus we can write formally

$$\int_X \mathcal{A}_{q'}^\alpha(g_K)(x)^{q'} \psi(x) d\mu(x) = \langle f_\psi, g_K \rangle, \quad (1.8)$$

where we define

$$f_\psi(y, t) := \begin{cases} M_{\alpha t} \psi(y) \overline{g_K(y, t)}^{q'/2} g_K(y, t)^{(q'/2)-1} & \text{when } g_K(y, t) \neq 0, \\ 0 & \text{when } g_K(y, t) = 0, \end{cases}$$

noting that  $g_K(y, t)^{(q'/2)-1}$  is not defined when  $g_K(y, t) = 0$  and  $q' < 2$ . However, the equality (1.8) is not valid until we show that  $f_\psi$  lies in  $T^{p, q, \alpha}(X)$ . To this end, estimate

$$\begin{aligned}
\mathcal{A}_q^\alpha(f_\psi) &\leq \left( \iint_{\Gamma^\alpha(x)} M_{\alpha t} \psi(y)^q |g_K(y, t)|^{q(q'-1)} \frac{d\mu(y)}{V(y, \alpha t)} \frac{dt}{t} \right)^{\frac{1}{q}} \\
&\leq \left( \iint_{\Gamma^\alpha(x)} \mathcal{M} \psi(x)^q |g_K(y, t)|^{q'} \frac{d\mu(y)}{V(y, \alpha t)} \frac{dt}{t} \right)^{\frac{1}{q}} \\
&= \mathcal{M} \psi(x) \mathcal{A}_{q'}^\alpha(g_K)(x)^{q'/q}.
\end{aligned}$$

Taking  $r$  such that  $1/p = 1/r + 1/(p'/q')'$  and using (HL), we then have

$$\begin{aligned}
\left\| \mathcal{A}_q^\alpha(f_\psi) \right\|_{L^p(X)} &\leq \left\| (\mathcal{M} \psi) \mathcal{A}_{q'}^\alpha(g_K)^{q'/q} \right\|_{L^p(X)} \\
&\leq \left\| \mathcal{M} \psi \right\|_{L^{(p'/q')'}(X)} \left\| \mathcal{A}_{q'}^\alpha(g_K)^{q'/q} \right\|_{L^r(X)} \\
&\lesssim_X \left\| \psi \right\|_{L^{(p'/q')'}(X)} \left\| \mathcal{A}_{q'}^\alpha(g_K) \right\|_{L^{r q'/q}(X)}^{q'/q} \\
&\leq \left\| \mathcal{A}_{q'}^\alpha(g_K) \right\|_{L^{r q'/q}(X)}^{q'/q}.
\end{aligned}$$

One can show that  $rq'/q = p'$ , and so  $f_\psi$  is in  $T^{p,q,\alpha}(X)$  by Lemma 1.3.3. By (1.8), taking the supremum over all  $\psi$  under consideration, we can write

$$\begin{aligned} \|g_K\|_{T^{p',q',\alpha}(X)}^{q'} &\leq \|\ell\| \|f_\psi\|_{T^{p,q,\alpha}(X)} \\ &\lesssim_X \|\ell\| \|g_K\|_{T^{p',q',\alpha}(X)}^{q'/q}, \end{aligned}$$

and consequently, using that  $\|g_K\|_{T^{p',q',\alpha}(X)} < \infty$ ,

$$\|g_K\|_{T^{p',q',\alpha}(X)} \lesssim_X \|\ell\|.$$

Since this estimate is independent of  $K$ , we have shown that  $g \in T^{p',q',\alpha}(X)$ , and therefore that  $(T^{p,q,\alpha}(X))^*$  is contained in  $T^{p',q',\alpha}(X)$ . This completes the proof when  $p < q$ .

To prove the statement for  $p > q$ , it suffices to show that the tent space  $T^{p',q',\alpha}(X)$  is reflexive. Thanks to the Eberlein–Šmulian theorem (see [1, Corollary 1.6.4]), this is equivalent to showing that every bounded sequence in  $T^{p',q',\alpha}(X)$  has a weakly convergent subsequence.

Let  $\{f_n\}_{n \in \mathbb{N}}$  be a sequence in  $T^{p',q',\alpha}(X)$  with  $\|f_n\|_{T^{p',q',\alpha}(X)} \leq 1$  for all  $n \in \mathbb{N}$ . Then by Lemma 1.3.3, for all cylindrical  $K \subset X^+$  the sequence  $\{f_n\}_{n \in \mathbb{N}}$  is bounded in  $L^{q'}(K)$ , and so by reflexivity of  $L^{q'}(K)$  we can find a subsequence  $\{f_{n_j}\}_{j \in \mathbb{N}}$  which converges weakly in  $L^{q'}(K)$ . We will show that this subsequence also converges weakly in  $T^{p',q',\alpha}(X)$ .

Let  $\ell \in (T^{p',q',\alpha}(X))^*$ . Since  $p' < q'$ , we have already shown that there exists a function  $g \in T^{p,q,\alpha}(X)$  such that  $\ell(f) = \langle f, g \rangle$ . For every  $\varepsilon > 0$ , we can find a cylindrical set  $K_\varepsilon \subset X^+$  such that

$$\|g - \mathbf{1}_{K_\varepsilon} g\|_{T^{p,q,\alpha}(X)} \leq \varepsilon.$$

Thus for all  $i, j \in \mathbb{N}$  and for all  $\varepsilon > 0$  we have

$$\begin{aligned} \ell(f_{n_i}) - \ell(f_{n_j}) &= \langle f_{n_i} - f_{n_j}, \mathbf{1}_{K_\varepsilon} g \rangle + \langle f_{n_i} - f_{n_j}, g - \mathbf{1}_{K_\varepsilon} g \rangle \\ &\leq \langle f_{n_i} - f_{n_j}, \mathbf{1}_{K_\varepsilon} g \rangle \\ &\quad + \left( \|f_{n_i}\|_{T^{p',q',\alpha}(X)} + \|f_{n_j}\|_{T^{p',q',\alpha}(X)} \right) \|g - \mathbf{1}_{K_\varepsilon} g\|_{T^{p,q,\alpha}(X)} \\ &\leq \langle f_{n_i} - f_{n_j}, \mathbf{1}_{K_\varepsilon} g \rangle + 2\varepsilon. \end{aligned}$$

As  $i, j \rightarrow \infty$ , the first term on the right hand side above tends to 0, and so we conclude that  $\{f_{n_j}\}_{j \in \mathbb{N}}$  converges weakly in  $T^{p',q',\alpha}(X)$ . This completes the proof.  $\square$

*Remark 1.3.11.* As mentioned earlier, property (HL) is weaker than doubling, but this is still a strong assumption. We note that for Proposition 1.3.10 to hold for a given pair  $(p, q)$ , the uncentred Hardy–Littlewood maximal operator need only be of strong type  $((p'/q')', (p'/q')')$ . Since  $(p'/q')'$  is increasing in  $p$  and decreasing in  $q$ , the condition required on  $X$  is stronger as  $p \rightarrow 1$  and  $q \rightarrow \infty$ .

Given Proposition 1.3.10, we can set up the vector-valued approach to tent spaces (first considered in [44]) using the method of [23]. Fix  $p \in (0, \infty)$ ,  $q \in (1, \infty)$ , and  $\alpha > 0$ . For simplicity of notation, write

$$L_\alpha^q(X^+) := L^q \left( X^+; \frac{d\mu(y)}{V(y, \alpha t)} \frac{dt}{t} \right).$$

We define an operator  $T_\alpha: T^{p,q,\alpha}(X) \rightarrow L^p(X; L_\alpha^q(X^+))$  from the tent space into the  $L_\alpha^q(X^+)$ -valued  $L^p$  space on  $X$  (see [35, §2] for vector-valued Lebesgue spaces) by setting

$$T_\alpha f(x)(y, t) := f(y, t) \mathbf{1}_{\Gamma^\alpha(x)}(y, t).$$

One can easily check that

$$\|T_\alpha f\|_{L^p(X; L_\alpha^q(X^+))} = \|f\|_{T^{p,q,\alpha}(X)},$$

and so the tent space  $T^{p,q,\alpha}(X)$  can be identified with its image under  $T_\alpha$  in the space  $L^p(X; L_\alpha^q(X^+))$ , provided that  $T_\alpha f$  is indeed a strongly measurable function of  $x \in X$ . This can be shown for  $q \in (1, \infty)$  by recourse to Pettis' measurability theorem [35, §2.1, Theorem 2], which reduces the question to that of weak measurability of  $T_\alpha f$ . To prove weak measurability, suppose  $g \in L_\alpha^{q'}(X)$ ; then

$$\langle T_\alpha f(x), g \rangle = \iint_{\Gamma^\alpha(x)} f(y, t) \overline{g(y, t)} \frac{d\mu(y)}{V(y, \alpha t)} \frac{dt}{t},$$

which is measurable in  $x$  by Lemma 1.4.6. Thus  $T_\alpha f$  is weakly measurable, and therefore  $T_\alpha f$  is strongly measurable as claimed.

Now assume  $p, q \in (1, \infty)$  and consider the operator  $\Pi_\alpha$ , sending  $X^+$ -valued functions on  $X$  to  $\mathbb{C}$ -valued functions on  $X^+$ , given by

$$(\Pi_\alpha F)(y, t) := \frac{1}{V(y, \alpha t)} \int_{B(y, \alpha t)} F(x)(y, t) d\mu(x)$$

whenever this expression is defined. Using the duality pairing from Proposition 1.3.10 and the duality pairing  $\langle \langle \cdot, \cdot \rangle \rangle$  for vector-valued  $L^p$  spaces, for  $f \in T^{p,q,\alpha}(X)$

and  $G \in L^{p'}(X; L_\alpha^{q'}(X^+))$  we have

$$\begin{aligned}
\langle \langle T_\alpha f, G \rangle \rangle &= \int_X \iint_{X^+} T_\alpha f(x)(y, t) \overline{G(x)(y, t)} \frac{d\mu(y)}{V(y, \alpha t)} \frac{dt}{t} d\mu(x) \\
&= \iint_{X^+} \frac{f(y, t)}{V(y, \alpha t)} \int_X \mathbf{1}_{B(y, \alpha t)}(x) \overline{G(x)(y, t)} d\mu(x) d\mu(y) \frac{dt}{t} \\
&= \iint_{X^+} f(y, t) \overline{(\Pi_\alpha G)(y, t)} d\mu(y) \frac{dt}{t} \\
&= \langle \langle f, \Pi_\alpha G \rangle \rangle.
\end{aligned}$$

Thus  $\Pi_\alpha$  maps  $L^{p'}(X; L_\alpha^{q'}(X^+))$  to  $T^{p', q', \alpha}(X)$ , by virtue of being the adjoint of  $T_\alpha$ . Consequently, the operator  $P_\alpha := T_\alpha \Pi_\alpha$  is bounded from  $L^p(X; L_\alpha^q(X^+))$  to itself for  $p, q \in (1, \infty)$ . A quick computation shows that  $\Pi_\alpha T_\alpha = I$ , so that  $P_\alpha$  projects  $L^p(X; L_\alpha^q(X^+))$  onto  $T_\alpha(T^{p, q, \alpha}(X))$ . This shows that  $T_\alpha(T^{p, q, \alpha}(X))$  is a complemented subspace of  $L^p(X; L_\alpha^q(X^+))$ . This observation leads to the basic interpolation result for tent spaces. Here  $[\cdot, \cdot]_\theta$  denotes the complex interpolation functor (see [22, Chapter 4]).

**Proposition 1.3.12.** *Suppose that  $X$  is either proper or doubling, and satisfies assumption (HL). Then for  $p_0, p_1, q_0$ , and  $q_1$  in  $(1, \infty)$ ,  $\theta \in [0, 1]$ , and  $\alpha > 0$ , we have (up to equivalence of norms)*

$$[T^{p_0, q_0, \alpha}(X), T^{p_1, q_1, \alpha}(X)]_\theta = T^{p, q, \alpha}(X),$$

where  $1/p = (1 - \theta)/p_0 + \theta/p_1$  and  $1/q = (1 - \theta)/q_0 + \theta/q_1$ .

*Proof.* Recall the identification

$$T^{r, s, \alpha}(X) \cong T_\alpha T^{r, s, \alpha}(X) \subset L^r(X; L_\alpha^s(X^+))$$

for all  $r \in (0, \infty)$  and  $s \in (1, \infty)$ . Since

$$\begin{aligned}
[L^{p_0}(X; L_\alpha^{q_0}(X^+)), L^{p_1}(X; L_\alpha^{q_1}(X^+))]_\theta &= L^p(X; [L_\alpha^{q_0}(X^+), L_\alpha^{q_1}(X^+)]_\theta) \\
&= L^p(X; L_\alpha^q(X^+))
\end{aligned}$$

applying the standard result on interpolation of complemented subspaces with common projections (see [89, Theorem 1.17.1.1]) yields

$$\begin{aligned}
[T^{p_0, q_0, \alpha}(X), T^{p_1, q_1, \alpha}(X)]_\theta &= L^p(X; L_\alpha^q(X^+)) \cap (T^{p_0, q_0, \alpha}(X) + T^{p_1, q_1, \alpha}(X)) \\
&= T^{p, q, \alpha}(X).
\end{aligned}$$

□

*Remark 1.3.13.* Since [89, Theorem 1.17.1.1] is true for any interpolation functor (not just complex interpolation), analogues of Proposition 1.3.12 hold for any interpolation functor  $F$  for which the spaces  $L^p(X; L_\alpha^q(X^+))$  form an appropriate interpolation scale. In particular, Proposition 1.3.12 (appropriately modified) holds for real interpolation.

*Remark 1.3.14.* Following the first submission of this article, the anonymous referee suggested a more direct proof of Proposition 1.3.12, which avoids interpolation of complemented subspaces. Since  $T_\alpha$  acts as an isometry both from  $T^{p_0, q_0, \alpha}(X)$  to  $L^{p_0}(X; L_\alpha^{q_0}(X^+))$  and from  $T^{p_1, q_1, \alpha}(X)$  to  $L^{p_1}(X; L_\alpha^{q_1}(X^+))$ , if  $f$  is in the interpolation space  $[T^{p_0, q_0, \alpha}(X), T^{p_1, q_1, \alpha}(X)]_\theta$ , then

$$\|f\|_{T^{p, q, \alpha}(X)} = \|T_\alpha f\|_{L^p(X; L_\alpha^q(X^+))} \leq \|f\|_{[T^{p_0, q_0, \alpha}(X), T^{p_1, q_1, \alpha}(X)]_\theta}$$

due to the exactness of the complex interpolation functor (and similarly for the real interpolation functor). Hence  $[T^{p_0, q_0, \alpha}(X), T^{p_1, q_1, \alpha}(X)]_\theta \subset T^{p, q, \alpha}(X)$ , and the reverse containment follows by duality. We have chosen to include both proofs for their own intrinsic interest.

## Endpoint results

We now consider the tent spaces  $T^{1, q, \alpha}(X)$  and  $T^{\infty, q, \alpha}(X)$ , and their relation to the rest of the tent space scale. In this section, we prove the following duality result using the method of [33].

**Proposition 1.3.15.** *Suppose  $X$  is doubling, and let  $\alpha > 0$  and  $q \in (1, \infty)$ . Then the pairing  $\langle \cdot, \cdot \rangle$  of Proposition 1.3.10 realises  $T^{\infty, q, \alpha}(X)$  as the Banach space dual of  $T^{1, q, \alpha}(X)$ , up to equivalence of norms.*

As in [33], we require a small series of definitions and lemmas to prove this result. We define *truncated cones* for  $x \in X$ ,  $\alpha, h > 0$  by

$$\Gamma_h^\alpha(x) := \Gamma^\alpha(x) \cap \{(y, t) \in X^+ : t < h\},$$

and corresponding Lusin operators for  $q > 0$  by

$$\mathcal{A}_q^\alpha(f|h)(x) := \left( \iint_{\Gamma_h^\alpha(x)} |f(y, t)|^q \frac{d\mu(y)}{V(y, \alpha t)} \frac{dt}{t} \right)^{\frac{1}{q}}.$$

One can show that  $\mathcal{A}_q^\alpha(f|h)$  is measurable in the same way as for  $\mathcal{A}_q^\alpha(f)$ .



**Lemma 1.3.16.** *For each measurable function  $g$  on  $X^+$ , each  $q \in [1, \infty)$ , and each  $M > 0$ , define*

$$h_{g,q,M}^\alpha(x) := \sup\{h > 0 : \mathcal{A}_q^\alpha(g|h)(x) \leq M\mathcal{C}_q^\alpha(g)(x)\}$$

*for  $x \in X$ . If  $X$  is doubling, then for sufficiently large  $M$  (depending on  $X$ ,  $q$ , and  $\alpha$ ), whenever  $B \subset X$  is a ball of radius  $r$ ,*

$$\mu\{x \in B : h_{g,q,M}^\alpha(x) \geq r\} \gtrsim_{X,\alpha} \mu(B).$$

*Proof.* Let  $B \subset X$  be a ball of radius  $r$ . Applying Lemmas 1.4.5 and 1.3.6, the definition of  $\mathcal{C}_q^\alpha$ , and doubling, we have

$$\begin{aligned} \int_B \mathcal{A}_q^\alpha(g|r)(x)^q d\mu(x) &= \int_B \iint_{\Gamma_r^\alpha(x)} \mathbf{1}_{T^\alpha((2\alpha+1)B)}(y,t) |g(y,t)|^q \frac{d\mu(y)}{V(y,\alpha t)} \frac{dt}{t} d\mu(x) \\ &\leq \int_B \iint_{\Gamma^\alpha(x)} \mathbf{1}_{T^\alpha((2\alpha+1)B)}(y,t) |g(y,t)|^q \frac{d\mu(y)}{V(y,\alpha t)} \frac{dt}{t} d\mu(x) \\ &\leq \iint_{T^\alpha((2\alpha+1)B)} |g(y,t)|^q d\mu(y) \frac{dt}{t} \\ &\leq \mu((2\alpha+1)B) \inf_{x \in B} \mathcal{C}_q^\alpha(g)(x)^q \\ &\lesssim_{X,\alpha} \mu(B) \inf_{x \in B} \mathcal{C}_q^\alpha(g)(x)^q. \end{aligned}$$

We can estimate

$$\begin{aligned} \int_B \mathcal{A}_q^\alpha(g|r)(x)^q d\mu(x) &\geq \left( M \inf_{x \in B} \mathcal{C}_q^\alpha(g)(x) \right)^q \cdot \\ &\quad \cdot \mu\left\{x \in B : \mathcal{A}_q^\alpha(g|r)(x) > M \inf_{x \in B} \mathcal{C}_q^\alpha(g)(x)\right\}, \end{aligned}$$

and after rearranging and combining with the previous estimate we get

$$M^q \left( \mu(B) - \mu\{x \in B : \mathcal{A}_q^\alpha(g|r)(x) \leq M \inf_{x \in B} \mathcal{C}_q^\alpha(g)(x)\} \right) \lesssim_{X,\alpha} \mu(B).$$

More rearranging and straightforward estimating yields

$$\mu\{x \in B : \mathcal{A}_q^\alpha(g|r)(x) \leq M\mathcal{C}_q^\alpha(g)(x)\} \geq (1 - M^{-q}C_{X,\alpha})\mu(B).$$

Since  $h_{g,q,M}^\alpha(x) \geq r$  if and only if  $\mathcal{A}_q^\alpha(g|r)(x) \leq M\mathcal{C}_q^\alpha(g)(x)$  as  $\mathcal{A}_q^\alpha(g|h)$  is increasing in  $h$ , we can rewrite this as

$$\mu\{x \in B : h_{g,q,M}^\alpha(x) \geq r\} \geq (1 - M^{-q}C_{X,\alpha})\mu(B).$$

Choosing  $M > C_{X,\alpha}^{1/q}$  completes the proof.  $\square$

**Corollary 1.3.17.** *With  $X$ ,  $g$ ,  $q$ , and  $\alpha$  as in the statement of the previous lemma, there exists  $M = M(X, q, \alpha)$  such that whenever  $\Phi$  is a nonnegative measurable function on  $X^+$ , we have*

$$\iint_{X^+} \Phi(y, t) V(y, \alpha t) d\mu(y) dt \lesssim_{X, \alpha} \int_X \iint_{\Gamma_{h_{g, q, M}^\alpha(x)/\alpha(x)}^\alpha} \Phi(y, t) d\mu(y) dt d\mu(x).$$

*Proof.* This is a straightforward application of Fubini–Tonelli’s theorem along with the previous lemma. Taking  $M$  sufficiently large, Lemma 1.3.16 gives

$$\begin{aligned} & \iint_{X^+} \Phi(y, t) V(y, \alpha t) d\mu(y) dt \\ & \lesssim_{X, \alpha} \iint_{X^+} \Phi(y, t) \int_{\{x \in B(y, \alpha t) : h_{g, q, M}^\alpha(x) \geq \alpha t\}} d\mu(x) d\mu(y) dt \\ & = \int_X \int_0^{h_{g, q, M}^\alpha(x)/\alpha} \int_{B(x, \alpha t)} \Phi(y, t) d\mu(y) dt d\mu(x) \\ & = \int_X \iint_{\Gamma_{h_{g, q, M}^\alpha(x)/\alpha(x)}^\alpha} \Phi(y, t) d\mu(y) dt d\mu(x) \end{aligned}$$

as required.  $\square$

We are now ready for the proof of the main duality result.

*Proof of Proposition 1.3.15.* First suppose  $f \in T^{1, q, \alpha}(X)$  and  $g \in T^{\infty, q', \alpha}(X)$ . By Corollary 1.3.17, there exists  $M = M(X, q, \alpha) > 0$  such that

$$\begin{aligned} & \iint_{X^+} |f(y, t)| |g(y, t)| d\mu(y) \frac{dt}{t} \\ & \lesssim_{X, \alpha} \int_X \iint_{\Gamma_{h(x)}^\alpha} |f(y, t)| |g(y, t)| \frac{d\mu(y)}{V(y, \alpha t)} \frac{dt}{t} d\mu(x), \end{aligned}$$

where  $h(x) := h_{g, q', M}^\alpha(x)/\alpha$ . Using Hölder’s inequality and the definition of  $h(x)$ , we find that

$$\begin{aligned} & \int_X \left( \iint_{\Gamma_{h(x)}^\alpha} |f(y, t)| |g(y, t)| \frac{d\mu(y)}{V(y, \alpha t)} \frac{dt}{t} \right) d\mu(x) \\ & \leq \int_X \mathcal{A}_q^\alpha(f|h(x))(x) \mathcal{A}_{q'}^\alpha(g|h(x))(x) d\mu(x) \\ & \leq M \int_X \mathcal{A}_q^\alpha(f)(x) \mathcal{C}_{q'}^\alpha(g)(x) d\mu(x) \\ & \lesssim_{X, q, \alpha} \|f\|_{T^{1, q, \alpha}(X)} \|g\|_{T^{\infty, q, \alpha}(X)}. \end{aligned}$$

Hence every  $g \in T^{\infty, q', \alpha}(X)$  induces a bounded linear functional on  $T^{1, q, \alpha}(X)$  via the pairing  $\langle f, g \rangle$  above, and so  $T^{\infty, q', \alpha}(X) \subset (T^{1, q, \alpha}(X))^*$ .

Conversely, suppose  $\ell \in (T^{1, q, \alpha}(X))^*$ . Then as in the proof of Proposition 1.3.10, from  $\ell$  we construct a function  $g \in L_{\text{loc}}^{q'}(X^+)$  such that

$$\iint_{X^+} f(y, t) \overline{g(y, t)} d\mu(y) \frac{dt}{t} = \ell(f)$$

for all  $f \in T^{1, q, \alpha}(X)$  with cylindrical support. We just need to show that  $g$  is in  $T^{\infty, q', \alpha}(X)$ . By the definition of the  $T^{\infty, q', \alpha}(X)$  norm, it suffices to estimate

$$\left( \frac{1}{\mu(B)} \iint_{T^\alpha(B)} |g(y, t)|^{q'} d\mu(y) \frac{dt}{t} \right)^{\frac{1}{q'}},$$

where  $B \subset X$  is an arbitrary ball.

For all nonnegative  $\psi \in L^q(T^\alpha(B))$  with  $\|\psi\|_{L^q(T^\alpha(B))} \leq 1$ , using that

$$S^\alpha(T^\alpha(B)) = B$$

we have that

$$\begin{aligned} \|\psi\|_{T^{1, q, \alpha}(X)} &= \int_B \mathcal{A}_q^\alpha(\psi)(x) d\mu(x) \\ &\leq \mu(B)^{1/q'} \|\psi\|_{T^{q, q, \alpha}(X)} \\ &= \mu(B)^{1/q'} \|\psi\|_{L^q(X^+)} \\ &\leq \mu(B)^{1/q'}. \end{aligned}$$

In particular,  $\psi$  is in  $T^{1, q, \alpha}(X)$ , so we can write

$$\iint_{T^\alpha(B)} \bar{g}\psi d\mu \frac{dt}{t} = \ell(\psi).$$

Arguing by duality and using the above computation, we then have

$$\begin{aligned} \left( \frac{1}{\mu(B)} \iint_{T^\alpha(B)} |g(y, t)|^{q'} d\mu(y) \frac{dt}{t} \right)^{1/q'} &= \mu(B)^{-1/q'} \sup_{\psi} \iint_{T^\alpha(B)} \bar{g}\psi d\mu \frac{dt}{t} \\ &= \mu(B)^{-1/q'} \sup_{\psi} \ell(\psi) \\ &\leq \mu(B)^{-1/q'} \|\ell\| \|\psi\|_{T^{1, q, \alpha}(X)} \\ &\leq \|\ell\|, \end{aligned}$$

where the supremum is taken over all  $\psi$  described above. Now taking the supremum over all balls  $B \subset X$ , we find that

$$\|g\|_{T^{\infty, q', \alpha}(X)} \leq \|\ell\|,$$

which completes the proof that  $(T^{1, q, \alpha}(X))^* \subset T^{\infty, q', \alpha}(X)$ .  $\square$

Once Proposition 1.3.15 is established, we can obtain the full scale of interpolation using the ‘convex reduction’ argument of [23, Theorem 3] and Wolff’s reiteration theorem (see [92] and [54]).

**Proposition 1.3.18.** *Suppose that  $X$  is doubling. Then for  $p_0, p_1 \in [1, \infty]$  (not both equal to  $\infty$ ),  $q_0$  and  $q_1$  in  $(1, \infty)$ ,  $\theta \in [0, 1]$ , and  $\alpha > 0$ , we have (up to equivalence of norms)*

$$[T^{p_0, q_0, \alpha}(X), T^{p_1, q_1, \alpha}(X)]_\theta = T^{p, q, \alpha}(X),$$

where  $1/p = (1 - \theta)/p_0 + \theta/p_1$  and  $1/q = (1 - \theta)/q_0 + \theta/q_1$ .

*Proof.* First we will show that

$$[T^{1, q_0, \alpha}(X), T^{p_1, q_1, \alpha}(X)]_\theta \supset T^{p, q, \alpha}(X). \quad (1.9)$$

Suppose  $f \in T^{p, q, \alpha}(X)$  is a cylindrically supported simple function. Then there exists another cylindrically supported simple function  $g$  such that  $f = g^2$ . Then

$$\|f\|_{T^{p, q, \alpha}(X)} = \|g\|_{T^{2p, 2q, \alpha}(X)}^2,$$

and so  $g$  is in  $T^{2p, 2q, \alpha}(X)$ . By Proposition 1.3.12 we have the identification

$$T^{2p, 2q, \alpha}(X) = [T^{2, 2q_0, \alpha}(X), T^{2p_1, 2q_1, \alpha}(X)]_\theta \quad (1.10)$$

up to equivalence of norms, and so by the definition of the complex interpolation functor (see Section 1.4.3), there exists for each  $\varepsilon > 0$  a function

$$G_\varepsilon \in \mathcal{F}(T^{2, 2q_0, \alpha}(X), T^{2p_1, 2q_1, \alpha}(X))$$

such that  $G_\varepsilon(\theta) = g$  and

$$\begin{aligned} \|G_\varepsilon\|_{\mathcal{F}(T^{2, 2q_0, \alpha}(X), T^{2p_1, 2q_1, \alpha}(X))} &\leq (1 + \varepsilon) \|g\|_{[T^{2, 2q_0, \alpha}(X), T^{2p_1, 2q_1, \alpha}(X)]_\theta} \\ &\simeq (1 + \varepsilon) \|g\|_{T^{2p, 2q, \alpha}(X)}, \end{aligned}$$

the implicit constant coming from the norm equivalence (1.10). Define  $F_\varepsilon := G_\varepsilon^2$ . Then we have

$$F_\varepsilon \in \mathcal{F}(T^{1, q_0, \alpha}(X), T^{p_1, q_1, \alpha}(X)),$$

with

$$\begin{aligned} \|F_\varepsilon\|_{\mathcal{F}(T^{1, q_0, \alpha}(X), T^{p_1, q_1, \alpha}(X))} &= \|G_\varepsilon\|_{\mathcal{F}(T^{2, 2q_0, \alpha}(X), T^{2p_1, 2q_1, \alpha}(X))}^2 \\ &\lesssim (1 + \varepsilon)^2 \|g\|_{T^{2p, 2q, \alpha}(X)}^2 \\ &= (1 + \varepsilon)^2 \|f\|_{T^{p, q, \alpha}(X)}. \end{aligned}$$

Therefore

$$\|f\|_{[T^{1,q_0,\alpha}(X), T^{p_1,q_1,\alpha}(X)]_\theta} \lesssim \|f\|_{T^{p,q,\alpha}(X)},$$

and so the inclusion (1.9) follows from the fact that cylindrically supported simple functions are dense in  $T^{p,q,\alpha}(X)$ .

By the duality theorem [22, Corollary 4.5.2] for interpolation (using reflexivity of  $T^{p_1,q_1,\alpha}(X)$ ), the inclusion (1.9), and Propositions 1.3.10 and 1.3.15, we have

$$[T^{p'_1,q'_1,\alpha}(X), T^{\infty,q'_0,\alpha}(X)]_{1-\theta} \subset T^{p',q',\alpha}(X).$$

Therefore we have the containment

$$[T^{p_0,q_0,\alpha}(X), T^{\infty,q_1,\alpha}(X)]_\theta \subset T^{p,q,\alpha}(X). \quad (1.11)$$

The reverse containment can be obtained from

$$[T^{1,q_0,\alpha}(X), T^{p_1,q_1,\alpha}(X)]_\theta \subset T^{p,q,\alpha}(X) \quad (1.12)$$

(for  $p_1, q_0, q_1 \in (1, \infty)$ ) by duality. The containment (1.12) can be obtained as in Remark 1.3.14, with  $p_0 = 1$  not changing the validity of this method.<sup>8</sup>

Finally, it remains to consider the case when  $p_0 = 1$  and  $p_1 = \infty$ . This is covered by Wolff reiteration. Set  $A_1 = T^{1,q_0,\alpha}(X)$ ,  $A_2 = T^{p,q,\alpha}(X)$ ,  $A_3 = T^{p+1,q_3,\alpha}(X)$ , and  $A_4 = T^{\infty,q_1,\alpha}(X)$  for an appropriate choice of  $q_3$ .<sup>9</sup> Then for an appropriate index  $\eta$ , we have  $[A_1, A_3]_{\theta/\eta} = A_2$  and  $[A_2, A_4]_{(\eta-\theta)/(1-\theta)} = A_3$ . Therefore by Wolff reiteration, we have  $[A_1, A_4]_\theta = A_2$ ; that is,

$$[T^{1,q_0,\alpha}(X), T^{\infty,q_1,\alpha}(X)]_\theta = T^{p,q,\alpha}(X).$$

This completes the proof. □

*Remark 1.3.19.* Note that doubling is not explicitly used in the above proof; it is only required to the extent that it is needed to prove Propositions 1.3.10 and 1.3.15 (as Proposition 1.3.12 follows from 1.3.10). If these propositions could be proven under some assumptions other than doubling, then it would follow that Proposition 1.3.18 holds under these assumptions.

*Remark 1.3.20.* The proof of [33, Lemma 5], which amounts to proving the containment (1.9), contains a mistake which is seemingly irreparable without resorting to more advanced techniques. This mistake appears on page 323, line

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<sup>8</sup>We thank the anonymous referee once more for this suggestion.

<sup>9</sup>More precisely, we need to take  $1/q_3 = (1 - 1/p')/q_0 + (1/p')/q_1$ .

-3, when it is stated that “ $A(f_k)$  is supported in  $O_k^* - O_{k+1}$ ” (and in particular, that  $A(f_k)$  is supported in  $O_{k+1}^c$ ). However (reverting to our notation), since  $f_k := \mathbf{1}_{T((O_k)_\gamma^*) \setminus T((O_{k+1})_\gamma^*)} f$ ,  $\mathcal{A}_2^1(f_k)$  is supported on

$$S^1(T((O_k)_\gamma^*) \setminus T((O_{k+1})_\gamma^*)) = (O_k)_\gamma^*$$

and we cannot conclude that  $\mathcal{A}_2^1(f_k)$  is supported away from  $O_{k+1}$ . Simple 1-dimensional examples can be constructed which show that this is false in general. Hence the containment (1.9) is not fully proven in [33]; the first valid proof in the Euclidean case that we know of is in [23] (the full range of interpolation is not obtained in [44].)

### 1.3.3 Change of aperture

Under the doubling assumption, the change of aperture result can be proven without assuming (NI) by means of the vector-valued method. The proof is a combination of the techniques of [44] and [23].

**Proposition 1.3.21.** *Suppose  $X$  is doubling. For  $\alpha, \beta \in (0, \infty)$  and  $p, q \in (0, \infty)$ , the tent space (quasi-)norms  $\|\cdot\|_{T^{p,q,\alpha}(X)}$  and  $\|\cdot\|_{T^{p,q,\beta}(X)}$  are equivalent.*

*Proof.* First suppose  $p, q \in (1, \infty)$ . Since  $X$  is doubling, we can replace our definition of  $\mathcal{A}_q^\alpha$  with the definition

$$\mathcal{A}_q^\alpha(f)(x)^q := \iint_{\Gamma^\alpha(x)} |f(y, t)|^q \frac{d\mu(y)}{V(y, t)} \frac{dt}{t};$$

using the notation of Section 1.3.1, this is the definition with  $\mathbf{a} = y$  and  $\mathbf{b} = 1$ . Having made this change, the vector-valued approach to tent spaces (see Section 1.3.2) transforms as follows. The tent space  $T^{p,q,\alpha}(X)$  now embeds isometrically into  $L^p(X; L_1^q(X^+))$  via the operator  $T_\alpha$  defined, as before, by

$$T_\alpha f(x)(y, t) := f(y, t) \mathbf{1}_{\Gamma^\alpha(x)}(y, t)$$

for  $f \in T^{p,q,\alpha}(X)$ . The adjoint of  $T_\alpha$  is the operator  $\Pi_\alpha$ , now defined by

$$(\Pi_\alpha G)(y, t) := \frac{1}{V(y, t)} \int_{B(y, \alpha t)} G(z)(y, t) d\mu(z)$$

for  $G \in L^p(X; L_1^q(X^+))$ . The composition  $P_\alpha := T_\alpha \Pi_\alpha$  is then a bounded projection from  $L^p(X; L_1^q(X^+))$  onto  $T_\alpha T^{p,q,\alpha}(X)$ , and can be written in the form

$$P_\alpha G(x)(y, t) = \frac{\mathbf{1}_{\Gamma^\alpha(x)}(y, t)}{V(y, t)} \int_{B(y, \alpha t)} G(z)(y, t) d\mu(z).$$

For  $f \in T^{p,q,\alpha}(X)$ , we can easily compute

$$P_\beta T_\alpha f(x)(y, t) = T_\beta f(x)(y, t) \frac{V(y, \min(\alpha, \beta)t)}{V(y, t)}. \quad (1.13)$$

Without loss of generality, suppose  $\beta > \alpha$ . Then we obviously have

$$\|\cdot\|_{T^{p,q,\alpha}(X)} \lesssim_{q,\alpha,\beta,X} \|\cdot\|_{T^{p,q,\beta}(X)}$$

by Remark 1.3.7. It remains to show that

$$\|\cdot\|_{T^{p,q,\beta}(X)} \lesssim_{p,q,\alpha,\beta,X} \|\cdot\|_{T^{p,q,\alpha}(X)}. \quad (1.14)$$

From (1.13) and doubling, for  $f \in T^{p,q,\alpha}(X)$  we have that

$$T_\beta f(x)(y, t) \lesssim_{X,\alpha} P_\beta T_\alpha f(x)(y, t),$$

and so we can write

$$\begin{aligned} \|f\|_{T^{p,q,\beta}(X)} &= \|T_\beta f\|_{L^p(X; L_1^q(X^+))} \\ &\lesssim_{X,\alpha} \|P_\beta T_\alpha f\|_{L^p(X; L_1^q(X^+))} \\ &\leq \|P_\beta\|_{\mathcal{L}(L^p(X; L_1^q(X^+)))} \|T_\alpha f\|_{L^p(X; L_1^q(X^+))} \\ &\lesssim_{p,q,\beta,X} \|f\|_{T^{p,q,\alpha}(X)} \end{aligned}$$

since  $P_\beta$  is a bounded operator on  $L^p(X; L_1^q(X^+))$ . This shows (1.14), and completes the proof for  $p, q \in (1, \infty)$ .

Now suppose that at least one of  $p$  and  $q$  is not in  $(1, \infty)$ , and suppose  $f \in T^{p,q,\alpha}(X)$  is a cylindrically supported simple function. Choose an integer  $M$  such that both  $Mp$  and  $Mq$  are in  $(1, \infty)$ . Then there exists a cylindrically supported simple function  $g$  with  $g^M = f$ . We then have

$$\begin{aligned} \|f\|_{T^{p,q,\alpha}(X)}^{1/M} &= \|g^M\|_{T^{p,q,\alpha}(X)}^{1/M} \\ &= \|g\|_{T^{Mp,Mq,\alpha}(X)} \\ &\simeq_{p,q,\alpha,\beta,X} \|g\|_{T^{Mp,Mq,\beta}(X)} \\ &= \|f\|_{T^{p,q,\beta}(X)}^{1/M}, \end{aligned}$$

and so the result is true for cylindrically supported simple functions, with an implicit constant which does not depend on the support of such a function. Since the cylindrically supported simple functions are dense in  $T^{p,q,\alpha}(X)$ , the proof is complete.  $\square$

*Remark 1.3.22.* Written more precisely, with  $p, q \in (0, \infty)$  and  $\beta < 1$ , the inequality (1.14) is of the form

$$\|\cdot\|_{T^{p,q,1}(X)} \lesssim_{p,q,X} \sup_{(y,t) \in X^+} \left( \frac{V(y,t)}{V(y,\beta t)} \right)^M \|\cdot\|_{T^{p,q,\beta}(X)}.$$

where  $M$  is such that  $Mp, Mq \in (1, \infty)$ .

### 1.3.4 Relations between $\mathcal{A}$ and $\mathcal{C}$

Again, this proposition follows from the methods of [33].

**Proposition 1.3.23.** *Suppose  $X$  satisfies (HL), and suppose  $0 < q < p < \infty$  and  $\alpha > 0$ . Then*

$$\|\mathcal{C}_q^\alpha(f)\|_{L^p(X)} \lesssim_{p,q,X} \|\mathcal{A}_q^\alpha(f)\|_{L^p(X)}.$$

*Proof.* Let  $B \subset X$  be a ball. Then by Fubini–Tonelli’s theorem, and using that  $S^\alpha(T^\alpha(B)) = B$ ,

$$\begin{aligned} & \frac{1}{\mu(B)} \iint_{T^\alpha(B)} |f(y,t)|^q d\mu(y) \frac{dt}{t} \\ &= \frac{1}{\mu(B)} \iint_{T^\alpha(B)} \frac{|f(y,t)|^q}{V(y,\alpha t)} \int_{B(y,\alpha t)} d\mu(x) d\mu(y) \frac{dt}{t} \\ &= \frac{1}{\mu(B)} \int_X \iint_{T^\alpha(B)} \mathbf{1}_{B(y,\alpha t)}(x) |f(y,t)|^q \frac{d\mu(y)}{V(y,\alpha t)} \frac{dt}{t} d\mu(x) \\ &= \frac{1}{\mu(B)} \int_B \iint_{T^\alpha(B)} \mathbf{1}_{B(x,\alpha t)}(y) |f(y,t)|^q \frac{d\mu(y)}{V(y,\alpha t)} \frac{dt}{t} d\mu(x) \\ &\leq \frac{1}{\mu(B)} \int_B \iint_{X^+} \mathbf{1}_{B(x,\alpha t)}(y) |f(y,t)|^q \frac{d\mu(y)}{V(y,\alpha t)} \frac{dt}{t} d\mu(x) \\ &= \frac{1}{\mu(B)} \int_B \mathcal{A}_q^\alpha(f)(x)^q d\mu(x). \end{aligned}$$

Now fix  $x \in X$  and take the supremum of both sides of this inequality over all balls  $B$  containing  $x$ . We find that

$$\mathcal{C}_q^\alpha(f)(x)^q \leq \mathcal{M}(\mathcal{A}_q^\alpha(f)^q)(x).$$

Since  $p/q > 1$ , we can apply (HL) to get

$$\begin{aligned} \|\mathcal{C}_q^\alpha(f)\|_{L^p(X)} &\leq \|\mathcal{M}(\mathcal{A}_q^\alpha(f)^q)^{1/q}\|_{L^p(X)} \\ &= \|\mathcal{M}(\mathcal{A}_q^\alpha(f)^q)\|_{L^{p/q}(X)}^{1/q} \\ &\lesssim_{p,q,X} \|\mathcal{A}_q^\alpha(f)^q\|_{L^{p/q}(X)}^{1/q} \\ &= \|\mathcal{A}_q^\alpha(f)\|_{L^p(X)} \end{aligned}$$



as desired.  $\square$

*Remark 1.3.24.* If  $X$  is doubling, and if  $p, q \in (0, \infty)$ , then for  $\alpha > 0$  we also have that

$$\left\| \mathcal{A}_q^\alpha(f) \right\|_{L^p(X)} \lesssim_{p,q,X} \left\| \mathcal{C}_q^\alpha(f) \right\|_{L^p(X)}.$$

This can be proven as in [33, §6], completely analogously to the proofs above.

## 1.4 Appendix: Assorted lemmas and notation

### 1.4.1 Tents, cones, and shadows

**Lemma 1.4.1.** *Suppose  $A$  and  $B$  are subsets of  $X$ , with  $A$  open, and suppose  $T^\alpha(A) \subset T^\alpha(B)$ . Then  $A \subset B$ .*

*Proof.* Suppose  $x \in A$ . Then  $\text{dist}(x, A^c) > 0$  since  $A$  is open, and so  $\text{dist}(x, A^c) > \alpha t$  for some  $t > 0$ . Hence  $(x, t) \in T^\alpha(A) \subset T^\alpha(B)$ , so that  $\text{dist}(x, B^c) > \alpha t > 0$ . Therefore  $x \in B$ .  $\square$

**Lemma 1.4.2.** *Let  $C \subset X^+$  be cylindrical, and suppose  $\alpha > 0$ . Then  $S^\alpha(C)$  is bounded.*

*Proof.* Write  $C \subset B(x, r) \times (a, b)$  for some  $x \in X$  and  $r, a, b > 0$ . Then  $S^\alpha(C) \subset S^\alpha(B(x, r) \times (a, b))$ , and one can easily show that

$$S^\alpha(B(x, r) \times (a, b)) \subset B(x, r + \alpha b),$$

showing the boundedness of  $S^\alpha(C)$ .  $\square$

**Lemma 1.4.3.** *Let  $C \subset X^+$ , and suppose  $\alpha > 0$ . Then  $T^\alpha(S^\alpha(C))$  is the minimal  $\alpha$ -tent containing  $C$ , in the sense that  $T^\alpha(S) \supset C$  for some  $S \subset X$  implies that  $T^\alpha(S^\alpha(C)) \subset T^\alpha(S)$ .*

*Proof.* A straightforward set-theoretic manipulation shows that  $C$  is contained in  $T^\alpha(S^\alpha(C))$ . We need to show that  $S^\alpha(C)$  is minimal with respect to this property.

Suppose that  $S \subset X$  is such that  $C \subset T^\alpha(S)$ , and suppose  $(w, t_w)$  is in  $T^\alpha(S^\alpha(C))$ . With the aim of showing that  $\text{dist}(w, S^c) > \alpha t_w$ , suppose that  $y \in S^c$ . Then  $\Gamma^\alpha(y) \cap T^\alpha(S) = \emptyset$ , and so  $\Gamma^\alpha(y) \cap C = \emptyset$  since  $T^\alpha(S)$  contains  $C$ . Thus  $y \in S^\alpha(C)^c$ , and so

$$d(w, y) \geq \text{dist}(w, S^\alpha(C)^c) > \alpha t_w$$

since  $(w, t_w) \in T^\alpha(S^\alpha(C))$ . Taking an infimum over  $y \in S^c$ , we get that

$$\text{dist}(w, S^c) > \alpha t_w,$$

which says precisely that  $(w, t_w)$  is in  $T^\alpha(S)$ . Therefore  $T^\alpha(S^\alpha(C)) \subset T^\alpha(S)$  as desired.  $\square$

**Lemma 1.4.4.** *For a cylindrical subset  $K \subset X^+$ , define*

$$\begin{aligned}\beta_0(K) &:= \inf_{B \subset X} \{\mu(B) : T^\alpha(B) \cap K \neq \emptyset\} \quad \text{and} \\ \beta_1(K) &:= \inf_{B \subset X} \{\mu(B) : T^\alpha(B) \supset K\},\end{aligned}$$

*with both infima taken over the set of balls  $B$  in  $X$ . Then  $\beta_1(K)$  is positive, and if  $X$  is proper or doubling, then  $\beta_0(K)$  is also positive.*

*Proof.* We first prove that  $\beta_0 := \beta_0(K)$  is positive, assuming that  $X$  is proper or doubling. Write

$$K \subset \overline{C} := \overline{B(x_0, r_0)} \times [a_0, b_0]$$

for some  $x_0 \in X$  and  $a_0, b_0, r_0 > 0$ . If  $B$  is a ball such that  $T^\alpha(B) \cap K \neq \emptyset$ , then we must have  $T^\alpha(B) \cap \overline{C} \neq \emptyset$ , and so we can estimate

$$\beta_0 \geq \inf_{B \subset X} \{\mu(B) : T^\alpha(B) \cap \overline{C} \neq \emptyset\}.$$

Note that if  $B = B(c(B), r(B))$  is a ball with  $c(B) \in \overline{B(x_0, r_0)}$ , then  $T^\alpha(B) \cap \overline{C} \neq \emptyset$  if and only if  $r(B) \geq \alpha a_0$ . Defining

$$I(x) := \inf\{V(x, r) : r > 0, T^\alpha(B(x, r)) \cap \overline{C} \neq \emptyset\}$$

for  $x \in X$ , we thus see that  $I(x) = V(x, \alpha a_0)$  when  $x \in \overline{B(x_0, r_0)}$ , and so  $I|_{\overline{B(x_0, r_0)}}$  is lower semicontinuous as long as the volume function is lower semicontinuous.

Now suppose  $B = B(y, \rho)$  is any ball with  $T^\alpha(B) \cap \overline{C} \neq \emptyset$ . Let  $(z, t_z)$  be a point in  $T^\alpha(B) \cap \overline{C}$ . We claim that the ball

$$\tilde{B} := B\left(z, \frac{1}{2}(\rho - d(z, y) + \alpha t_z)\right)$$

is contained in  $B$ , centred in  $\overline{B(x_0, r_0)}$ , and is such that  $T^\alpha(\tilde{B}) \cap \overline{C} \neq \emptyset$ . The second fact is obvious:  $(z, t_z) \in \overline{C}$  implies  $z \in \overline{B(x_0, r_0)}$ . For the first fact, observe that

$$\begin{aligned}\tilde{B} &\subset B(y, d(z, y) + (\rho - d(z, y) + \alpha t_z)/2) \\ &= B(y, (\rho + d(z, y) + \alpha t_z)/2) \\ &\subset B(y, (\rho + (\rho - \alpha t_z) + \alpha t_z)/2) \\ &= B(y, \rho),\end{aligned}$$

since  $(z, t_z) \in T^\alpha(B)$  implies that  $d(z, y) < \rho - \alpha t_z$ . Finally, we have  $(z, t_z) \in T^\alpha(\tilde{B})$ : since  $c(\tilde{B}) = z$ , we just need to show that  $t_z < r(\tilde{B})/\alpha$ . Indeed, we have

$$\frac{r(\tilde{B})}{\alpha} = \frac{1}{2} \left( \frac{\rho - d(z, y)}{\alpha} + t_z \right),$$

and  $t_z < (\rho - d(z, y))/\alpha$  as above.

The previous paragraph shows that

$$\inf_{x \in X} I(x) \geq \inf_{x \in \overline{B(x_0, r_0)}} I(x),$$

and so we are reduced to showing that the right hand side of this inequality is positive, since  $\beta_0 \geq \inf_{x \in X} I(x)$ .

**If  $X$  is proper:** Since  $\overline{B(x_0, r_0)}$  is compact and  $I|_{\overline{B(x_0, r_0)}}$  is lower semicontinuous,  $I|_{\overline{B(x_0, r_0)}}$  attains its infimum on  $\overline{B(x_0, r_0)}$ . That is,

$$\inf_{x \in \overline{B(x_0, r_0)}} I(x) = \min_{x \in \overline{B(x_0, r_0)}} I_x > 0, \quad (1.15)$$

by positivity of the ball volume function.

**If  $X$  is doubling:** Since  $I(x) = V(x, \alpha a_0)$  when  $x \in \overline{B(x_0, r_0)}$ , we can write

$$\inf_{x \in \overline{B(x_0, r_0)}} I(x) \geq \inf_{x \in \overline{B(x_0, r_0)}} V(x, \varepsilon),$$

where  $\varepsilon = \min(\alpha a_0, 3r_0)$ . If  $x \in \overline{B(x_0, r_0)}$ , then  $\overline{B(x_0, r_0)} \subset \overline{B(x, 2r_0)} \subset B(x, 3r_0)$ , and so since  $3r_0/\varepsilon \geq 1$ ,

$$\begin{aligned} V(x_0, r_0) &\leq V(x, 3r_0) \\ &= V(x, \varepsilon(3r_0/\varepsilon)) \\ &\lesssim_X V(x, \varepsilon). \end{aligned}$$

Hence  $V(x, \varepsilon) \gtrsim_X V(x_0, r_0)$ , and therefore

$$\inf_{x \in \overline{B(x_0, r_0)}} V(x, \varepsilon) \gtrsim V(x_0, r_0) > 0 \quad (1.16)$$

as desired.

We now prove that  $\beta_1 = \beta_1(K)$  is positive. Recall from Lemma 1.4.3 that if  $T^\alpha(B) \supset K$ , then  $T^\alpha(B) \supset T^\alpha(S^\alpha(K))$ . Since shadows are open, Lemma 1.4.1 tells us that  $B \supset S^\alpha(K)$ . Hence  $\mu(B) \geq \mu(S^\alpha(K))$ , and so

$$\beta_1 \geq \mu(S^\alpha(K)) > 0$$

by positivity of the ball volume function.<sup>10</sup> □

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<sup>10</sup>If  $S^\alpha(K)$  is a ball, then  $\beta_1(K) = \mu(S^\alpha(K))$ .

**Lemma 1.4.5.** *Let  $B$  be an open ball in  $X$  of radius  $r$ . Then for all  $x \in B$ , the truncated cone  $\Gamma_r^\alpha(x)$  is contained in  $T^\alpha((2\alpha + 1)B)$ .*

*Proof.* Suppose  $(y, t) \in \Gamma_r^\alpha(x)$  and  $z \in ((2\alpha + 1)B)^c$ , so that  $d(y, x) < \alpha t < \alpha r$  and  $d(c(B), z) \geq (2\alpha + 1)r$ . Then by the triangle inequality

$$\begin{aligned} d(y, z) &\geq d(c(B), z) - d(c(B), x) - d(x, y) \\ &> (2\alpha + 1)r - r - \alpha r \\ &= \alpha r \\ &> \alpha t, \end{aligned}$$

so that  $\text{dist}(y, ((2\alpha + 1)B)^c) > \alpha t$ , which yields  $(y, t) \in T^\alpha((2\alpha + 1)B)$ .  $\square$

## 1.4.2 Measurability

We assume  $(X, d, \mu)$  has the implicit assumptions from Section 1.2.

**Lemma 1.4.6.** *Let  $\alpha > 0$ , and suppose  $\Phi$  is a non-negative measurable function on  $X^+$ . Then the function*

$$g: x \mapsto \iint_{\Gamma^\alpha(x)} \Phi(y, t) d\mu(y) \frac{dt}{t}$$

*is  $\mu$ -measurable.*

We present two proofs of this lemma: one uses an abstract measurability result, while the other is elementary (and in fact stronger, proving that  $g$  is not only measurable but lower semicontinuous).

*First proof.* By [67, Theorem 3.1], it suffices to show that the function

$$F(x, (y, t)) := \mathbf{1}_{B(y, \alpha t)}(x) \Phi(y, t)$$

is measurable on  $X \times X^+$ . For  $\varepsilon > 0$ , define

$$f_\varepsilon(x, (y, t)) := \frac{\text{dist}(x, \overline{B(y, \alpha t)})}{\text{dist}(x, \overline{B(y, \alpha t)}) + \text{dist}(x, B(y, \alpha t + \varepsilon)^c)}.$$

Then  $f_\varepsilon(x, (y, t))$  is continuous in  $x$ , and converges pointwise to  $\mathbf{1}_{B(y, \alpha t)}(x)$  as  $\varepsilon \rightarrow 0$ . Hence

$$F(x, (y, t)) = \lim_{\varepsilon \rightarrow 0} f_\varepsilon(x, (y, t)) \Phi(y, t) =: \lim_{\varepsilon \rightarrow 0} F_\varepsilon(x, (y, t)),$$

and therefore it suffices to show that each  $F_\varepsilon(x, (y, t))$  is measurable on  $X \times X^+$ . Since  $F_\varepsilon$  is continuous in  $x$  and measurable in  $(y, t)$ ,  $F_\varepsilon$  is measurable on  $X \times X^+$ ,<sup>11</sup> and the proof is complete.  $\square$

*Second proof.* For all  $x \in X$  and  $\varepsilon > 0$ , define the vertically translated cone

$$\Gamma_\varepsilon^\alpha(x) := \{(y, t) \in X^+ : (y, t - \varepsilon) \in \Gamma^\alpha(x)\} \subset \Gamma^\alpha(x).$$

If  $y \in B(x, \alpha\varepsilon)$ , then it is easy to show that  $\Gamma_\varepsilon^\alpha(x) \subset \Gamma^\alpha(y)$ : indeed, if  $(z, t) \in \Gamma_\varepsilon^\alpha(x)$ , then  $d(z, x) < \alpha(t - \varepsilon)$ , and so

$$d(z, y) \leq d(z, x) + d(x, y) < \alpha(t - \varepsilon) + \alpha\varepsilon = \alpha t.$$

For all  $x \in X$  and  $\varepsilon > 0$ , define

$$g_\varepsilon(x) := \iint_{\Gamma_\varepsilon^\alpha(x)} \Phi(y, t) d\mu(y) \frac{dt}{t}.$$

For each  $x \in X$ , as  $\varepsilon \searrow 0$ , we have  $g_\varepsilon(x) \nearrow g(x)$  by monotone convergence. Fix  $\lambda > 0$ , and suppose that  $g(x) > \lambda$ . Then there exists  $\varepsilon(x)$  such that  $g_{\varepsilon(x)}(x) > \lambda$ . If  $y \in B(x, \alpha\varepsilon(x))$ , then by the previous paragraph we have

$$g(y) \geq g_{\varepsilon(x)}(x) > \lambda.$$

Therefore  $g$  is lower semicontinuous, and thus measurable.  $\square$

**Lemma 1.4.7.** *Let  $f$  be a measurable function on  $X^+$ ,  $q \in (0, \infty)$ , and  $\alpha > 0$ . Then  $\mathcal{C}_q^\alpha(f)$  is lower semicontinuous.*

*Proof.* Let  $\lambda > 0$ , and suppose  $x \in X$  is such that  $\mathcal{C}_q^\alpha(f)(x) > \lambda$ . Then there exists a ball  $B \ni x$  such that

$$\frac{1}{\mu(B)} \iint_{T^\alpha(B)} |f(y, t)|^q d\mu(y) \frac{dt}{t} > \lambda^q.$$

Hence for any  $z \in B$ , we have  $\mathcal{C}_q^\alpha(f)(z) > \lambda$ , and so the set  $\{x \in X : \mathcal{C}_q^\alpha(f)(x) > \lambda\}$  is open.  $\square$

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<sup>11</sup>See [41, Theorem 1], which tells us that  $F_\varepsilon$  is Lusin measurable; this implies Borel measurability on  $X \times X^+$ .

### 1.4.3 Interpolation

Here we fix some notation involving complex interpolation.

An *interpolation pair* is a pair  $(B_0, B_1)$  of complex Banach spaces which admit embeddings into a single complex Hausdorff topological vector space. To such a pair we can associate the Banach space  $B_0 + B_1$ , endowed with the norm

$$\|x\|_{B_0+B_1} := \inf\{\|x_0\|_{B_0} + \|x_1\|_{B_1} : x_0 \in B_0, x_1 \in B_1, x = x_0 + x_1\}.$$

We can then consider the space  $\mathcal{F}(B_0, B_1)$  of functions  $f$  from the closed strip

$$\overline{S} = \{z \in \mathbb{C} : 0 \leq \operatorname{Re}(z) \leq 1\}$$

into the Banach space  $B_0 + B_1$ , such that

- $f$  is analytic on the interior of  $S$  and continuous on  $\overline{S}$ ,
- $f(z) \in B_j$  whenever  $\operatorname{Re}(z) = j$  ( $j \in \{0, 1\}$ ), and
- the traces  $f_j := f|_{\operatorname{Re} z = j}$  ( $j \in \{0, 1\}$ ) are continuous maps into  $B_j$  which vanish at infinity.

The space  $\mathcal{F}(B_0, B_1)$  is a Banach space when endowed with the norm

$$\|f\|_{\mathcal{F}(B_0, B_1)} := \max \left( \sup_{\operatorname{Re} z = 0} \|f(z)\|_{B_0}, \sup_{\operatorname{Re} z = 1} \|f(z)\|_{B_1} \right).$$

We define the *complex interpolation space*  $[B_0, B_1]_\theta$  for  $\theta \in [0, 1]$  to be the subspace of  $B_0 + B_1$  defined by

$$[B_0, B_1]_\theta := \{f(\theta) : f \in \mathcal{F}(B_0, B_1)\}$$

endowed with the norm

$$\|x\|_{[B_0, B_1]_\theta} := \inf_{f(\theta)=x} \|f\|_{\mathcal{F}(B_0, B_1)}.$$

# Chapter 2

## Non-uniformly local tent spaces

This article is joint work with Mikko Kemppainen.

### Abstract

We develop a theory of ‘non-uniformly local’ tent spaces on metric measure spaces. As our main result, we give a remarkably simple proof of the atomic decomposition.

### 2.1 Introduction

The theory of global tent spaces on Euclidean space was first considered by Coifman, Meyer, and Stein [33], and has since become a central framework for understanding Hardy spaces defined by square functions. Upon replacing Euclidean space with a doubling metric measure space, the theory is largely unchanged. Details of this generalisation can be found in [3], although this was known to harmonic analysts for some time.

Tent spaces on Riemannian manifolds with doubling volume measure were used by Auscher, McIntosh, and Russ in [13], where a ‘first order approach’ to Hardy spaces associated with the Laplacian  $-\Delta$  (or more accurately, the corresponding Hodge–Dirac operator) was investigated. A corresponding *local* tent space theory, now on manifolds with *exponentially locally doubling* volume measure, was considered by Carbonaro, McIntosh, and Morris [30], with applications to operators such as  $-\Delta + a$  for  $a > 0$ . The locality arises from the ‘spectral gap’ between 0 and  $\sigma(-\Delta + a) \subset [a, \infty)$  and means that the relevant information of a function can be captured from small time diffusion, which in turn allows one to exploit the locally doubling nature of the manifold under investigation.

Hence the related tent spaces consist of functions of space-time variables  $(y, t)$  with  $0 < t < 1$  instead of  $0 < t < \infty$ .

The motivation for non-uniformly local tent spaces comes from the setting of Gaussian harmonic analysis, in which one considers the *Ornstein–Uhlenbeck operator*  $L = -\Delta + x \cdot \nabla$  on  $\mathbb{R}^n$  equipped with the usual Euclidean distance and the Gaussian measure  $d\gamma(x) = (2\pi)^{-n/2} e^{-|x|^2/2} dx$ . Here  $\sigma(L) = \{0, 1, 2, \dots\}$ , but despite the evident spectral gap, one cannot make use of a uniformly local tent space because the rapidly decaying measure  $\gamma$  is non-doubling. This was remedied by Maas, van Neerven, and Portal [64], who defined the ‘Gaussian tent spaces’  $\mathfrak{t}^p(\gamma)$  to consist of functions on the region  $D = \{(y, t) \in \mathbb{R}^n \times (0, \infty) : t < m(y)\}$ . Here  $m(y) = \min(1, |y|^{-1})$  is the *admissibility function* of Mauceri and Meda [68], who showed that  $\gamma$  is doubling on the family of ‘admissible balls’  $B(x, t)$  with  $t \leq m(x)$ . In [78], Portal then defined the ‘Gaussian Hardy space’  $\mathfrak{h}^1(\gamma)$  using the conical square function

$$Su(x) = \left( \int_0^{2m(x)} \int_{B(x,t)} |t \nabla e^{-t^2 L} u(y)|^2 d\gamma(y) \frac{dt}{t} \right)^{1/2},$$

and showed that the Riesz transform  $\nabla L^{-1/2}$  is bounded from  $\mathfrak{h}^1(\gamma)$  to  $L^1(\gamma)$ . This relied on the atomic decomposition on  $\mathfrak{t}^1(\gamma)$ , which was established in [64], along with a square function estimate from [63]. The Gaussian Hardy space is also known to interpolate with  $L^2(\gamma)$ , in the sense that  $[\mathfrak{h}^1(\gamma), L^2(\gamma)]_\theta = L^p(\gamma)$  for  $1/p = 1 - \theta/2$  [77]. Note that dimension-independent boundedness of  $\nabla L^{-1/2}$  on  $L^p(\gamma)$  for  $1 < p < \infty$  is a classical result of Meyer [74].

Our long-term aim is to generalise this theory to the setting where, given an appropriate ‘potential function’  $\phi$  on a Riemannian manifold  $X$  (or some more general space) with volume measure  $\mu$ , one considers the *Witten Laplacian*  $L = -\Delta + \nabla \phi \cdot \nabla$  equipped with the geodesic distance and the measure  $d\gamma = e^{-\phi} d\mu$ . An admissibility function can then be defined by  $m(x) = \min(1, |\nabla \phi(x)|^{-1})$ , with a suitable interpretation of  $\nabla$  if  $\phi$  is not differentiable, and the setting of Gaussian harmonic analysis is recovered by taking  $X = \mathbb{R}^n$  and  $\phi(x) = \frac{n}{2} \log(2\pi) + \frac{|x|^2}{2}$ . The Riesz transform associated with the Witten Laplacian has been studied for instance by Bakry in [20], where  $L^p(\gamma)$  boundedness for  $1 < p < \infty$  is proven under a  $\phi$ -related curvature assumption.

In this article we define and study the corresponding local tent spaces  $\mathfrak{t}^{p,q}(\gamma)$ . Our main result is the atomic decomposition Theorem 2.4.5. This allows us to identify the dual of  $\mathfrak{t}^{1,q}(\gamma)$  with the local tent space  $\mathfrak{t}^{\infty,q'}(\gamma)$ , and to show that the local tent spaces form a complex interpolation scale. In Appendix 2.6 we prove a



‘cone covering lemma’ for non-negatively curved Riemannian manifolds. It gives a stronger version of Lemma 2.4.4 that is applicable also in the vector-valued theory of tent spaces (see [59, 60]).

A different approach to Gaussian Hardy spaces was introduced in [68], where the *atomic* Hardy space  $H^1(\gamma)$  was introduced. This theory has also been extended to certain metric measure spaces (see [28, 29]). While many interesting singular integral operators, such as imaginary powers of the Ornstein–Uhlenbeck operator, have been shown to act boundedly from  $H^1(\gamma)$  to  $L^1(\gamma)$  (see [68]), it should be noted that this is not the case for the Riesz transform (see [69]). This marks the crucial difference between the atomic Hardy space  $H^1(\gamma)$  and  $\mathfrak{h}^1(\gamma)$ .

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## 2.2 Weighted measures and admissible balls

We begin by formulating the abstract framework in which we develop our theory. Let  $(X, d, \mu)$  be a metric measure space: that is, a metric space  $(X, d)$  equipped with a Borel measure  $\mu$ . We assume that every ball  $B \subset X$  comes with a given center  $c_B$  and a radius  $r_B > 0$ , and that the volume  $\mu(B)$  is finite and nonzero. Furthermore, we assume that the metric space  $(X, d)$  is geometrically doubling: that is, we assume that there exists a natural number  $N \geq 1$  such that for every ball  $B \subset X$  of radius  $r_B$ , there exist at most  $N$  mutually disjoint balls of radius  $r_B/2$  contained in  $B$ .

Given a measurable real-valued function  $\phi$  on  $X$ , we consider the weighted measure

$$d\gamma(x) := e^{-\phi(x)} d\mu(x).$$

Furthermore, we fix a function  $m: X \rightarrow (0, \infty)$ , which we call an *admissibility function*. For every  $\alpha > 0$ , this defines the family of *admissible balls*

$$\mathcal{B}_\alpha := \{B \subset X : 0 < r_B \leq \alpha m(c_B)\}.$$

These objects are required to satisfy the following doubling condition:

- (A) For every  $\alpha > 0$ ,  $\gamma$  is doubling on  $\mathcal{B}_\alpha$ , in the sense that there exists a constant  $C_\alpha \geq 1$  such that for all  $\alpha$ -admissible balls  $B \in \mathcal{B}_\alpha$ ,

$$\gamma(2B) \leq C_\alpha \gamma(B).$$

Here and in what follows, we write  $\lambda B = B(c_B, \lambda r_B)$  for the expansion of a ball  $B$  by  $\lambda \geq 1$ .

*Remark 2.2.1.* Condition (A) implies that for every  $\alpha > 0$  and every  $\lambda \geq 1$ , there exists a constant  $C_{\alpha,\lambda} \geq 1$  such that for all  $\alpha$ -admissible balls  $B \in \mathcal{B}_\alpha$ ,

$$\gamma(\lambda B) \leq C_{\alpha,\lambda} \gamma(B). \quad (2.1)$$

We now describe two classes of examples of  $\phi$  and  $m$ .

**Example 2.2.2** (Distance functions). Assume that the underlying measure  $\mu$  is doubling, let  $\Omega \subset X$  be a measurable set of ‘origins’, and let  $a, a' > 0$ . Define  $\phi$  by

$$\phi(x) := a + a' \operatorname{dist}(x, \Omega)^2.$$

An admissibility function can then be defined by

$$m(x) = \min \left( 1, \frac{1}{\operatorname{dist}(x, \Omega)} \right).$$

Taking  $X$  to be  $\mathbb{R}^n$  (equipped with the usual Euclidean distance and Lebesgue measure),  $\Omega = \{0\}$ , and  $(a, a') = (n \log(2\pi)/2, 1/2)$ , we recover the setting of Gaussian harmonic analysis.

**Claim 2.2.3.** *Condition (A) is satisfied with  $C_\alpha = D_\mu e^{a'\alpha(5\alpha+6)}$ , where  $D_\mu$  is the doubling constant of the underlying measure  $\mu$ .*

*Proof.* Since  $\mu$  is doubling, it suffices to show that for every  $\alpha$ -admissible ball  $B \in \mathcal{B}_\alpha$  we have

$$\begin{cases} e^{-\phi(x)} \leq C'_\alpha e^{-\phi(c_B)} & \text{when } x \in 2B, \text{ and} \\ e^{-\phi(x)} \geq C''_\alpha e^{-\phi(c_B)} & \text{when } x \in B. \end{cases} \quad (2.2)$$

Indeed, this would imply that

$$\gamma(2B) = \int_{2B} e^{-\phi(x)} d\mu(x) \leq C'_\alpha \mu(2B) e^{-\phi(c_B)}$$

and

$$\gamma(B) = \int_B e^{-\phi(x)} d\mu(x) \geq C''_\alpha \mu(B) e^{-\phi(c_B)},$$

so that

$$\frac{\gamma(2B)}{\gamma(B)} \leq \frac{C'_\alpha \mu(2B)}{C''_\alpha \mu(B)} \leq C_\alpha := D_\mu \frac{C'_\alpha}{C''_\alpha}.$$

To see that the first inequality in (2.2) holds with  $C'_\alpha = e^{4a'\alpha(\alpha+1)}$ , observe that if  $x \in 2B$ , then

$$\text{dist}(c_B, \Omega) \leq 2\alpha m(x) + \text{dist}(x, \Omega).$$

Indeed, if  $\text{dist}(c_B, \Omega) \geq \text{dist}(x, \Omega)$ , then  $m(c_B) \leq m(x)$ , and so

$$\text{dist}(c_B, \Omega) \leq d(c_B, x) + \text{dist}(x, \Omega) \leq 2\alpha m(c_B) + \text{dist}(x, \Omega) \leq 2\alpha m(x) + \text{dist}(x, \Omega).$$

Consequently we have

$$\text{dist}(c_B, \Omega)^2 \leq 4\alpha m(x)^2 + 4\alpha m(x) \text{dist}(x, \Omega) + \text{dist}(x, \Omega)^2 \leq 4\alpha^2 + 4\alpha + \text{dist}(x, \Omega)^2,$$

and so

$$e^{-a' \text{dist}(x, \Omega)^2} \leq e^{4a'\alpha(\alpha+1)} e^{-a' \text{dist}(c_B, \Omega)^2}.$$

Similarly, the second inequality in (2.2) with  $C''_\alpha = e^{-a'\alpha(\alpha+2)}$  follows after noting that if  $x \in B$ , then

$$\text{dist}(x, \Omega) \leq d(x, c_B) + \text{dist}(c_B, \Omega) \leq \alpha m(c_B) + \text{dist}(c_B, \Omega).$$

Thus

$$\text{dist}(x, \Omega)^2 \leq \alpha^2 + 2\alpha + \text{dist}(c_B, \Omega)^2$$

and

$$e^{-a' \text{dist}(x, \Omega)^2} \geq e^{-a'\alpha(\alpha+2)} e^{-a' \text{dist}(c_B, \Omega)^2}.$$

Putting these estimates together, we have

$$C_\alpha = D_\mu e^{4a'\alpha(\alpha+1)} e^{a'\alpha(\alpha+2)} = D_\mu e^{a'\alpha(5\alpha+6)}$$

as claimed. □

**Example 2.2.4** ( $C^2$  potentials). In this example, let  $(X, g)$  be a connected Riemannian manifold ( $C^2$  is sufficient) with doubling volume measure, let  $\phi \in C^2(X)$ , and assume that the following condition is satisfied:

(B) there exists a constant  $M > 0$  such that for every unit speed geodesic  $\rho: [0, \ell] \rightarrow X$ , we have

$$|(\phi \circ \rho)''(t)| \leq M|(\phi \circ \rho)'(t)| \quad (2.3)$$

for all  $t \in (0, \ell)$  such that  $|(\phi \circ \rho)'(t)| > 1$ .

Alternatively, we can assume the following inequivalent condition, which is neater but generally harder to verify:

(H) there exists a constant  $M > 0$  such that

$$||\text{Hess } \phi(x)|| \leq M|\nabla \phi(x)| \quad (2.4)$$

for all  $x \in X$  such that  $|\nabla \phi(x)| > 1$ .

Note that (B) can be interpreted as a one-dimensional version of (H); indeed, when  $X$  is one-dimensional, both conditions are equivalent.

If either of the above conditions are satisfied, we define an admissibility function by

$$m(x) := \min \left( 1, \frac{1}{|\nabla \phi(x)|} \right)$$

for  $x \in X$ , with  $m(x) := 1$  when  $|\nabla \phi(x)| = 0$ .

**Claim 2.2.5.** *If  $d(x, y) \leq \alpha$  then  $m(x) \leq e^{M\alpha}m(y)$ .*

*Proof.* Here we assume condition (H); the proof under assumption (B) requires only a simple modification.

Given  $\varepsilon > 0$ , we first take a continuous arclength-parametrised path

$$\rho: [0, d(x, y) + \varepsilon] \rightarrow X$$

connecting  $x$  to  $y$  (we may take  $\varepsilon = 0$  when  $X$  is complete, and the argument is slightly simpler in this case). Since  $\phi$  is twice continuously differentiable, the function  $m_\rho := m \circ \rho$  is absolutely continuous on  $[0, d(x, y) + \varepsilon]$ , and hence differentiable almost everywhere on this interval. We compute the derivative of  $m_\rho(t)$  whenever  $m_\rho$  is differentiable. If  $t$  is such that  $|\nabla \phi(\rho(t))| \leq 1$  in a neighbourhood of  $t$ , then  $\partial_t m_\rho(t) = 0$ . If  $t$  is such that  $|\nabla \phi(\rho(t))| > 1$  in a neighbourhood of  $t$ , then

$$\partial_t m_\rho(t) = \partial_t (|\nabla \phi(\rho(t))|^{-1}) = \frac{-\partial_t |\nabla \phi(\rho(t))|}{|\nabla \phi(\rho(t))|^2}.$$

Using the estimate

$$|\partial_t |\nabla \phi(\rho(t))|| \leq ||\text{Hess } \phi(\rho(t))||$$

along with assumption (H), we find that

$$|\partial_t m_\rho(t)| \leq \frac{||\text{Hess } \phi(\rho(t))||}{|\nabla \phi(\rho(t))|^2} \leq \frac{M}{|\nabla \phi(\rho(t))|}$$

for all  $t$  such that  $m_\rho(t)$  is differentiable.

Since  $m_\rho(t)$  is differentiable almost everywhere, we have

$$\begin{aligned} |\log m_\rho(d(x, y)) - \log m_\rho(0)| &\leq \sup_{0 < t < d(x, y) + \varepsilon} |\partial_t \log m_\rho(t)| d(x, y) \\ &\leq \sup_{0 < t < d(x, y) + \varepsilon} |\partial_t \log m_\rho(t)| \alpha, \end{aligned}$$

where the supremum is taken over all  $t \in (0, d(x, y) + \varepsilon)$  such that  $m_\rho(t)$  is differentiable. Note that

$$|\partial_t \log m_\rho(t)| = \frac{|\partial_t m_\rho(t)|}{|m_\rho(t)|},$$

and so by the estimate above we have that

$$|\partial_t \log m_\rho(t)| \leq \frac{M}{|\nabla \phi(\rho(t))|} |\nabla \phi(\rho(t))| = M.$$

Therefore

$$|\log m_\rho(d(x, y) + \varepsilon) - \log m_\rho(0)| \leq M\alpha,$$

and so

$$e^{|\log(m(y)/m(x))|} \leq e^{M(\alpha + \varepsilon)} =: c'_\alpha e^{M\varepsilon}.$$

This holds for every  $\varepsilon > 0$ , so by taking the limit of both sides as  $\varepsilon \rightarrow 0$  we obtain

$$e^{|\log(m(y)/m(x))|} \leq c'_\alpha. \tag{2.5}$$

Without loss of generality, we can suppose that  $m(x) \geq m(y)$ . Then

$$|\log(m(y)/m(x))| = \log(m(x)/m(y)),$$

and (2.5) implies that

$$\frac{m(x)}{m(y)} \leq c'_\alpha,$$

which completes the proof. □

**Claim 2.2.6.** *Condition (A) is satisfied, with  $C_\alpha = D_\mu e^{3\alpha e^{M\alpha}}$ .*

*Proof.* As in the previous example, it suffices to show that for every  $B \in \mathcal{B}_\alpha$  we have

$$\begin{cases} e^{-\phi(x)} \leq C'_\alpha e^{-\phi(c_B)}, & \text{when } x \in 2B, \\ e^{-\phi(x)} \geq C''_\alpha e^{-\phi(c_B)}, & \text{when } x \in B. \end{cases} \quad (2.6)$$

This is implied (with  $C'_\alpha = e^{\alpha c'_\alpha}$  and  $C''_\alpha = e^{-2\alpha c'_\alpha}$ ) by the estimate

$$|\phi(x) - \phi(c_B)| \leq \lambda \alpha c'_\alpha \quad \forall x \in \lambda B,$$

for all  $\lambda \geq 1$  and  $x \in \lambda B$ , which we now show. If  $x \in \lambda B$ , then we have

$$|\phi(x) - \phi(c_B)| \leq \sup_{y \in \lambda B} |\nabla \phi(y)| d(x, c_B).$$

Since  $B$  is  $\alpha$ -admissible, for all  $x, y \in \lambda B$  Claim 2.2.5 yields

$$d(x, c_B) \leq \lambda r_B \leq \lambda \alpha m(c_B) \leq \lambda \alpha c'_\alpha m(y) \leq \lambda \alpha c'_\alpha |\nabla \phi(y)|^{-1},$$

and so  $|\phi(x) - \phi(c_B)| \leq \lambda \alpha c'_\alpha$ . As in the previous example, we then have

$$C_\alpha = D_\mu \frac{C'_\alpha}{C''_\alpha} = D_\mu e^{3\alpha c'_\alpha}.$$

Using  $c'_\alpha = e^{M\alpha}$  (from Claim 2.2.5) yields the result.  $\square$

For a concrete subexample, let  $(X, d, \mu)$  be the Euclidean space  $\mathbb{R}^n$  with the usual Euclidean distance and Lebesgue measure, and let  $\phi \in \mathbb{R}[x_1, \dots, x_n]$  be a polynomial. Condition (B) is easily verified, although condition (H) may not hold when  $n \geq 2$ . Taking  $\phi(x) = \frac{n \log(2\pi)}{2} + \frac{1}{2} \sum_{i=1}^n x_i^2$ , we again recover the setting of Gaussian harmonic analysis. However, in this case the constants  $c'_\alpha$  and  $C_\alpha$  have significantly worse  $\alpha$ -dependence than the constants we found in the previous example. This is because conditions (B) and (H) are less restrictive than assuming  $\phi$  is given in terms of a distance function.

*Remark 2.2.7.* The utility of an admissibility function is eventually judged by its applicability to the local Hardy space theory. More precisely, one needs to obtain suitable ‘error estimates’ in the spirit of [78, Section 5]. The only known example of such at the time of writing is the setting of  $\mathbb{R}^n$  with  $\phi(x) = \frac{n}{2} \log \pi + |x|^2$  and  $m(x) = \min(1, |x|^{-1})$ .

## 2.3 Local tent spaces: the reflexive range

We now introduce the main topic of the paper — the non-uniformly local tent spaces. Let  $\phi$  and  $m$  be given and satisfy (A) from Section 2.2. Denote the resulting weighted measure by  $\gamma$ .

**Definition 2.3.1.** Let  $0 < p, q < \infty$  and  $\alpha > 0$ . The *local tent space*  $\mathfrak{t}_\alpha^{p,q}(\gamma)$  is the set of all measurable functions  $f$  defined on the *admissible region*

$$D = \{(y, t) \in X \times (0, \infty) : t < m(y)\}$$

such that the functional

$$\mathcal{A}_q^\alpha f(x) = \left( \iint_{\Gamma_\alpha(x)} |f(y, t)|^q \frac{d\gamma(y)}{\gamma(B(y, t))} \frac{dt}{t} \right)^{1/q}$$

satisfies

$$\|f\|_{\mathfrak{t}_\alpha^{p,q}(\gamma)} := \|\mathcal{A}_q^\alpha f\|_{L^p(\gamma)} < \infty.$$

Here  $\Gamma_\alpha(x) = \{(y, t) \in D : d(x, y) < \alpha t\}$  is the *admissible cone* of aperture  $\alpha$  at  $x \in X$ .

It is clear that  $\|\cdot\|_{\mathfrak{t}_\alpha^{p,q}(\gamma)}$  is a norm on  $\mathfrak{t}_\alpha^{p,q}(\gamma)$  when  $p, q \in [1, \infty)$ , and a quasi-norm when  $p < 1$  or  $q < 1$ . Following the argument of [3, Proposition 3.4] with doubling replaced by local doubling, we can show that  $\mathfrak{t}_\alpha^{p,q}(\gamma)$  is complete in this (quasi-)norm.

*Remark 2.3.2.* The choice  $\phi = 0$  and  $m = \infty$  recovers the setting of global tent spaces [3], whereas  $\phi = 0$  and  $m = 1$  gives the setting of uniformly local tent spaces by Carbonaro, McIntosh and Morris [30].

For  $1 < p, q < \infty$ , the properties of  $\mathfrak{t}_\alpha^{p,q}(\gamma)$  can be studied, as in [44], by embedding the space into an  $L^p$ -space of  $L^q$ -valued functions. More precisely, let us write  $L^q(D)$  for the space of  $q$ -integrable functions on  $D$  with respect to the measure  $\frac{d\gamma(y) dt}{t\gamma(B(y, t))}$ , so that

$$J_\alpha : \mathfrak{t}_\alpha^{p,q}(\gamma) \hookrightarrow L^p(\gamma; L^q(D)), \quad J_\alpha f(x) = \mathbf{1}_{\Gamma_\alpha(x)} f$$

defines an isometry. We will show that  $J_\alpha$  embeds  $\mathfrak{t}_\alpha^{p,q}(\gamma)$  as a complemented subspace of  $L^p(\gamma; L^q(D))$ , with

$$N_\alpha U(x; y, t) = \mathbf{1}_{B(y, \alpha t)}(x) \int_{B(y, \alpha t)} U(z; y, t) d\gamma(z)$$

defining a bounded projection of  $L^p(\gamma; L^q(D))$  onto the image of  $\mathfrak{t}_\alpha^{p,q}(\gamma)$ , where  $U \in L^p(\gamma; L^q(D))$ ,  $x \in X$ , and  $(y, t) \in D$ .

To see that  $N_\alpha$  is bounded, we first observe that

$$\begin{aligned} |N_\alpha U(x; y, t)| &\leq \mathbf{1}_{B(y, \alpha t)}(x) \int_{B(y, \alpha t)} |U(z; y, t)| d\gamma(z) \\ &\leq \sup_{\substack{B \ni x \\ B \in \mathcal{B}_\alpha}} \int_B |U(z; y, t)| d\gamma(z) \\ &= \mathcal{M}_\alpha U(x; y, t), \end{aligned}$$

where  $\mathcal{M}_\alpha$  is the  $L^q(\Sigma)$ -valued  $\alpha$ -local maximal function from Appendix 2.5, with  $\Sigma = (D, \frac{d\gamma(y) dt}{t\gamma(B(y, t))})$ . Consequently,

$$\|N_\alpha U\|_{L^p(\gamma; L^q(D))} \leq \|\mathcal{M}_\alpha U\|_{L^p(\gamma; L^q(D))} \lesssim_{p,q} c_X C_{\alpha, c_X} \|U\|_{L^p(\gamma; L^2(D))},$$

(see Appendix 2.5).

An immediate consequence of this vector-valued approach is the following theorem, detailing the behaviour of the local tent spaces in the reflexive range.

**Theorem 2.3.3.** *Let  $1 < p, q < \infty$ . We have*

- (change of aperture)  $\|f\|_{\mathfrak{t}_\alpha^{p,q}(\gamma)} \asymp_{p,q,\alpha,\beta} \|f\|_{\mathfrak{t}_\beta^{p,q}(\gamma)}$  for  $0 < \beta < \alpha < \infty$ ,
- (duality)  $\mathfrak{t}_\alpha^{p,q}(\gamma)^* = \mathfrak{t}_\alpha^{p',q'}(\gamma)$ , realised by the duality pairing

$$\langle f, g \rangle = \iint_D f(y, t) \overline{g(y, t)} d\gamma(y) \frac{dt}{t},$$

- (complex interpolation)  $[\mathfrak{t}_\alpha^{p_0, q_0}(\gamma), \mathfrak{t}_\alpha^{p_1, q_1}(\gamma)]_\theta = \mathfrak{t}_\alpha^{p, q}(\gamma)$  when  $1 < p_0 \leq p_1 < \infty$  and  $1 < q_0 \leq q_1 < \infty$ , with  $1/p = (1-\theta)/p_0 + \theta/p_1$ ,  $1/q = (1-\theta)/q_0 + \theta/q_1$ .

*Proof.* For our claim on change of aperture, we follow [44] and begin by noting that for suitable  $f$  we have

$$N_\alpha J_\beta f(x; y, t) = \frac{\gamma(B(y, \beta t))}{\gamma(B(y, \alpha t))} J_\alpha f(x; y, t).$$

Then

$$\begin{aligned} \|f\|_{\mathfrak{t}_\alpha^{p,q}(\gamma)} &= \|J_\alpha f\|_{L^p(\gamma; L^q(D))} = \frac{\gamma(B(y, \alpha t))}{\gamma(B(y, \beta t))} \|N_\alpha J_\beta f\|_{L^p(\gamma; L^q(D))} \\ &\lesssim_{p,q} C_{\beta, \alpha/\beta} C_{\alpha, c_X} \|J_\beta f\|_{L^p(\gamma; L^q(D))} \\ &= C_{\beta, \alpha/\beta} C_{\alpha, c_X} \|f\|_{\mathfrak{t}_\beta^{p,q}(\gamma)}, \end{aligned}$$



where the constants are from Remark 2.2.1.

Now  $\mathfrak{t}_\alpha^{p,q}(\gamma)$  is embedded in  $L^p(\gamma; L^q(D))$  as the range of the projection  $N_\alpha$ , whose dual is isomorphic to the range of  $N_\alpha^*$  on  $L^p(\gamma; L^q(D))^* = L^{p'}(\gamma; L^{q'}(D))$ , which, in turn, is isometrically isomorphic to  $\mathfrak{t}_\alpha^{p',q'}(\gamma)$  (because  $N_\alpha^* = N_\alpha$ ). The duality is realised as

$$\begin{aligned} \langle f, g \rangle &= \langle J_\alpha f, J_\alpha g \rangle \\ &= \int_X \langle \mathbf{1}_{\Gamma_\alpha(x)} f, \mathbf{1}_{\Gamma_\alpha(x)} g \rangle d\gamma(x) \\ &= \int_X \iint_{\Gamma_\alpha(x)} f(y, t) \overline{g(y, t)} \frac{d\gamma(y)}{\gamma(B(y, t))} \frac{dt}{t} d\gamma(x) \\ &= \iint_D f(y, t) \overline{g(y, t)} d\gamma(y) \frac{dt}{t}. \end{aligned}$$

For  $1 < p_0 \leq p_1 < \infty$  and  $1 < q_0 \leq q_1 < \infty$  the interpolation of tent spaces follows, by the standard result on interpolation of complemented subspaces [89, Section 1.17], from the fact that

$$[L^{p_0}(\gamma; L^{q_0}(D)), L^{p_1}(\gamma; L^{q_1}(D))]_\theta = L^p(\gamma; L^q(D)).$$

□

*Remark 2.3.4.* The dependence on  $\alpha$  in the aperture change constant  $C_{1,\alpha}C_{\alpha,c_X}$  (between  $\mathfrak{t}_\alpha^{p,q}(\gamma)$  and  $\mathfrak{t}_1^{p,q}(\gamma)$ ) is not optimal in general. For instance, on  $(\mathbb{R}^n, dx)$ , the optimal dependence is  $\alpha^{n/\min(p,2)}$  (see [6]), while  $C_{1,\alpha}C_{\alpha,c_X} \asymp \alpha^n$ . Note however, that on  $(\mathbb{R}^n, \gamma)$  we have  $C_{1,\alpha}C_{\alpha,c_X} \lesssim e^{c\alpha^2}$  for some constant  $c$ . We return to this in Section 2.4.

The change of aperture and interpolation results extend to  $1 \leq p, q < \infty$  by a convex reduction due to Bernal ([23], see also [3]).

**Corollary 2.3.5.** *Let  $1 \leq q < \infty$ . We have*

- (change of aperture)  $\|f\|_{\mathfrak{t}_\alpha^{1,q}(\gamma)} \asymp_{q,\alpha,\beta} \|f\|_{\mathfrak{t}_\beta^{1,q}(\gamma)}$  for  $0 < \beta < \alpha < \infty$ ,
- (complex interpolation)  $[\mathfrak{t}_\alpha^{p_0,q_0}(\gamma), \mathfrak{t}_\alpha^{p_1,q_1}(\gamma)]_\theta = \mathfrak{t}_\alpha^{p,q}(\gamma)$  when  $1 \leq p_0 \leq p_1 < \infty$  and  $1 < q_0 \leq q_1 < \infty$ , with  $1/p = (1-\theta)/p_0 + \theta/p_1$ ,  $1/q = (1-\theta)/q_0 + \theta/q_1$ .

## 2.4 Endpoints: $\mathfrak{t}^{1,q}$ and $\mathfrak{t}^{\infty,q}$

In this section, under the assumption that the space  $X$  is complete, we study the endpoints of the local tent space scale: the spaces  $\mathfrak{t}_\alpha^{1,q}(\gamma)$  and  $\mathfrak{t}_\alpha^{\infty,q}(\gamma)$  (with

$1 \leq q < \infty$ ). In particular, employing Corollary 2.3.5 we prove, following the argument in [59], that elements of  $\mathfrak{t}_\alpha^{1,q}(\gamma)$  can be decomposed into ‘atoms’. From this we deduce duality, interpolation, and (quantified) change of aperture results for the full local tent space scale  $\mathfrak{t}_\alpha^{p,q}(\gamma)$  ( $1 \leq p \leq \infty$ ,  $1 \leq q < \infty$ ). We write  $\mathfrak{t}^{1,q} := \mathfrak{t}_1^{1,q}$  for notational simplicity. We do not consider  $q = \infty$ . As in [33], this requires additional continuity and convergence assumptions.

### 2.4.1 Atomic decomposition

Fix  $(X, d, \mu)$ ,  $\phi$ , and  $m$  as in the previous section. The *admissible tent*  $T(O)$  over an open set  $O \subset X$  is given by

$$T(O) := D \setminus \Gamma(O^c),$$

where  $\Gamma(O^c) := \cup_{x \in O^c} \Gamma(x)$ .

**Definition 2.4.1.** Fix  $\alpha > 0$  and  $q \geq 1$ . A function  $a$  on  $D$  is called an  $\alpha$ - $\mathfrak{t}^{1,q}$ -atom (or more succinctly, a  $\alpha$ -atom) if there exists an  $\alpha$ -admissible ball  $B \in \mathcal{B}_\alpha$  such that  $\text{supp } a \subset T(B)$  and

$$\iint_{T(B)} |a(y, t)|^q d\gamma(y) \frac{dt}{t} \leq \frac{1}{\gamma(B)^{q-1}}.$$

Observe that for such a function  $a$ ,

$$\|a\|_{\mathfrak{t}^{1,q}(\gamma)} = \int_B \mathcal{A}_q a(x)^q d\gamma(x) \leq \gamma(B)^{\frac{q-1}{q}} \left( \int_B \mathcal{A}_q a(x)^q d\gamma(x) \right)^{1/q} \lesssim 1.$$

Furthermore, if  $(a_k)_{k \in \mathbb{N}}$  is a sequence of  $\alpha$ - $\mathfrak{t}^{1,q}$ -atoms for some  $\alpha > 0$ , then the series  $f = \sum_k \lambda_k a_k$  converges in  $\mathfrak{t}^{1,q}(\gamma)$  when  $\sum_k |\lambda_k| < \infty$ . The *atomic tent space*  $\mathfrak{t}_{\text{at}}^{1,q}(\gamma)$  consisting of such functions  $f$  becomes a Banach space when normed by

$$\|f\|_{\mathfrak{t}_{\text{at}}^{1,q}(\gamma)} = \inf \left\{ \sum_k |\lambda_k| : f = \sum_k \lambda_k a_k \right\}.$$

**Lemma 2.4.2.** Suppose that  $E \subset X$  is a bounded open set. Then there exists a countable sequence of disjoint admissible balls  $B^j \subset E$  such that

$$T(E) \subset \bigcup_{j \geq 1} T(5B^j).$$

*Proof.* Let  $\delta_1 = \sup\{r_B : B \subset E \text{ admissible}\}$  and begin by choosing an admissible ball  $B^1 \subset E$  with radius  $r_1 > \delta_1/2$ . Proceeding inductively we put

$$\delta_{k+1} = \sup\{r_B : B \subset E \text{ admissible}, B \cap B^j = \emptyset, j = 1, \dots, k\}$$

and choose (if possible) an admissible ball  $B^{k+1} \subset E$  with radius  $r_{k+1} > \delta_{k+1}/2$  disjoint from  $B^1, \dots, B^k$ . Given a  $(y, t) \in T(E)$  we show that  $B(y, t) \subset 5B^j$  for some  $j$ . It is possible to pick the first index  $j$  for which  $B(y, t) \cap B^j$  is nonempty. Indeed, if on the contrary  $B(y, t)$  was disjoint from every  $B^j$ , then,  $B(y, t)$  being admissible and contained in  $E$ , we would have  $t \geq \delta_j$  for all  $j$  which under the assumption that  $(X, d)$  is geometrically doubling contradicts the boundedness of  $E$ . By construction, we have  $t \leq \delta_j \leq 2r_j$  and so  $B(y, t) \subset 5B^j$ , as required.  $\square$

*Remark 2.4.3.* The above lemma is a stronger version of a ‘local Vitali covering lemma’, which is otherwise identical but claims only that  $E \subset \bigcup_{j \geq 1} 5B^j$  without reference to tents (see also Remark 2.5.2).

The following lemma regarding pointwise estimates for  $\mathcal{A}$ -functionals, which appears implicitly in [33, Theorem 4], lies at the heart of our proof of the atomic decomposition. This is the only point at which we seem to need completeness; we suspect that this assumption can be removed or at least weakened.

**Lemma 2.4.4.** *Suppose  $X$  is complete, let  $q \geq 1$  and let  $f$  be a measurable function in  $D$ . Let  $\lambda > 0$  and write  $E = \{x \in X : \mathcal{A}_q^3 f(x) > \lambda\}$ . Then  $\mathcal{A}_q(f \mathbf{1}_{D \setminus T(E)})(x) \leq \lambda$  for all  $x \in X$ .*

*Proof.* If  $x \notin E$ , then  $\mathcal{A}_q(f \mathbf{1}_{D \setminus T(E)})(x) \leq \mathcal{A}_q^3 f(x) \leq \lambda$ .

If  $x \in E$ , then by completeness of  $X$  we can choose a point  $x_0 \in X \setminus E$  such that  $d(x, x_0) = d(x, X \setminus E)$ . We show that  $\Gamma(x) \setminus T(E) \subset \Gamma^3(x_0)$ : let  $(y, t) \in \Gamma(x) \setminus T(E)$  so that  $d(x, y) < t$  and  $B(y, t) \not\subset E$ . Now  $B(y, t) \subset B(x, 2t)$ , which means that  $B(x, 2t) \not\subset E$  and so  $x_0 \in B(x, 2t)$ . Moreover  $B(x, 2t) \subset B(y, 3t)$  so that  $(y, t) \in \Gamma^3(x_0)$ . Therefore  $\mathcal{A}_q(f \mathbf{1}_{D \setminus T(E)})(x) \leq \mathcal{A}_q^3 f(x_0) \leq \lambda$ .  $\square$

**Theorem 2.4.5.** *Suppose  $X$  is complete, and let  $q \geq 1$ . For every  $f \in \mathfrak{t}^{1,q}(\gamma)$ , there exist  $5\text{-}\mathfrak{t}^{1,q}$ -atoms  $a_k$  and scalars  $\lambda_k$  such that*

$$f = \sum_k \lambda_k a_k, \tag{2.7}$$

with

$$\sum_k |\lambda_k| \simeq \|f\|_{\mathfrak{t}^{1,q}(\gamma)}.$$

We call the series (2.7) an *atomic decomposition* of  $f$ .

*Proof.* We first derive atomic decompositions for the dense class of boundedly-supported functions in  $\mathfrak{t}^{1,q}(\gamma)$ , and then argue by completeness of  $\mathfrak{t}_{\text{at}}^{1,q}(\gamma)$ . Given a function  $f$  in  $\mathfrak{t}^{1,q}(\gamma)$  with bounded support, we consider the bounded open sets

$$E_k = \{x \in X : \mathcal{A}_q^3 f(x) > 2^k\}$$

for each integer  $k$ . Applying Lemma 2.4.2 to these sets provides us with disjoint balls  $B_k^j \subset E_k$  such that

$$T(E_k) \subset \bigcup_{j \geq 1} T(5B_k^j).$$

In addition, we take a collection of functions  $\chi_k^j$  (cf. [59, Theorem 11]) satisfying

$$0 \leq \chi_k^j \leq 1, \quad \sum_{j \geq 1} \chi_k^j = 1 \text{ on } T(E_k), \quad \text{and} \quad \text{supp } \chi_k^j \subset T(5B_k^j).$$

Writing  $A_k := T(E_k) \setminus T(E_{k+1})$ , we can decompose  $f$  as

$$f = \sum_{k \in \mathbb{Z}} \mathbf{1}_{A_k} f = \sum_{k \in \mathbb{Z}} \sum_{j \geq 1} \chi_k^j \mathbf{1}_{A_k} f = \sum_{k \in \mathbb{Z}} \sum_{j \geq 1} \lambda_k^j a_k^j,$$

where

$$\lambda_k^j = \gamma(5B_k^j)^{1/q'} \left( \int_{5B_k^j} \mathcal{A}_q(f \mathbf{1}_{A_k})(x)^q d\gamma(x) \right)^{1/q}.$$

Observe that  $a_k^j = \chi_k^j \mathbf{1}_{A_k} f / \lambda_k^j$  is a 5-atom supported in  $T(5B_k^j)$ .

What remains is to control the sum of the scalars  $\lambda_k^j$ . By Lemma 2.4.4, we have

$$\mathcal{A}_q(f \mathbf{1}_{A_k})(x) \leq \mathcal{A}_q(f \mathbf{1}_{D \setminus T(E_{k+1})})(x) \leq 2^{k+1}$$

for all  $x \in X$ , and so

$$\lambda_k^j \leq \gamma(5B_k^j)^{1/q'} 2^{k+1}.$$

Consequently,

$$\sum_{k \in \mathbb{Z}} \sum_{j \geq 1} \lambda_k^j \leq \sum_{k \in \mathbb{Z}} 2^{k+1} \sum_{j \geq 1} \gamma(5B_k^j) \lesssim \sum_{k \in \mathbb{Z}} 2^{k+1} \gamma(E_k) \lesssim \|\mathcal{A}_q^3 f\|_{L^1(\gamma)} \lesssim \|f\|_{\mathfrak{t}^{1,q}(\gamma)},$$

where the last step follows by Corollary 2.3.5.

We have thus shown that  $\|f\|_{\mathfrak{t}_{\text{at}}^{1,q}(\gamma)} \approx \|f\|_{\mathfrak{t}^{1,q}(\gamma)}$  for boundedly supported  $f$  in  $\mathfrak{t}^{1,q}(\gamma)$ . Since the class of such functions is dense in  $\mathfrak{t}^{1,q}(\gamma)$ , the completeness of  $\mathfrak{t}_{\text{at}}^{1,q}(\gamma)$  guarantees that every  $f \in \mathfrak{t}^{1,q}(\gamma)$  has an atomic decomposition.  $\square$

*Remark 2.4.6.* Maas, van Neerven and Portal established the above result in the setting of Gaussian  $\mathbb{R}^n$  by a different method, which relies on Gaussian Whitney decompositions [64, Theorem 3.4]. In addition, they showed that decompositions into  $\alpha$ -atoms exist for every  $\alpha > 1$  [64, Lemma 3.6]. Such a result may not hold in this level of generality due to the lack of geometric information.

### 2.4.2 Duality, interpolation and change of aperture

We present three corollaries of the atomic decomposition theorem, which holds when  $X$  is complete.

The dual of  $\mathfrak{t}^{1,q}(\gamma)$  can be identified with the space  $\mathfrak{t}^{\infty,q'}(\gamma)$ , consisting of those functions  $g$  on  $D$  for which

$$\|g\|_{\mathfrak{t}^{\infty,q'}(\gamma)} = \sup_{B \in \mathcal{B}_5} \left( \frac{1}{\gamma(B)} \iint_{T(B)} |g(y,t)|^{q'} d\gamma(y) \frac{dt}{t} \right)^{1/q'}$$

is finite. Note that we take a supremum over 5-admissible balls, reflecting the fact that we have atomic decompositions of elements of  $\mathfrak{t}^{1,q}(\gamma)$  into 5-atoms. For the reader's convenience, we present the standard proof, following [33, Theorem 1 (b)].

**Corollary 2.4.7.** *Suppose  $X$  is complete, and let  $q \geq 1$ . Then the pairing*

$$\langle f, g \rangle = \iint_D f(y,t) \overline{g(y,t)} d\gamma(y) \frac{dt}{t}, \quad f \in \mathfrak{t}^{1,q}(\gamma), \quad g \in \mathfrak{t}^{\infty,q'}(\gamma), \quad (2.8)$$

*realises  $\mathfrak{t}^{\infty,q'}(\gamma)$  as the dual of  $\mathfrak{t}^{1,q}(\gamma)$ .*

*Proof.* To see that (2.8) defines a bounded linear functional on  $\mathfrak{t}^{1,q}(\gamma)$  for every  $g \in \mathfrak{t}^{\infty,q'}(\gamma)$ , it suffices (by Theorem 2.4.5) to test the pairing on atoms. For any atom  $a$  associated with a ball  $B \in \mathcal{B}_5$  we have

$$\begin{aligned} |\langle a, g \rangle| &\leq \iint_{T(B)} |a\bar{g}| d\gamma \frac{dt}{t} \\ &\leq \left( \iint_{T(B)} |a|^q d\gamma \frac{dt}{t} \right)^{1/q} \left( \iint_{T(B)} |g|^{q'} d\gamma \frac{dt}{t} \right)^{1/q'} \\ &\leq \|g\|_{\mathfrak{t}^{\infty,q'}(\gamma)}. \end{aligned}$$

To show that every functional  $\Lambda \in \mathfrak{t}^{1,q}(\gamma)^*$  arises in this way, we first note that each  $f \in L^q(T(B))$ , with  $B \in \mathcal{B}_5$ , satisfies

$$\|f\|_{\mathfrak{t}^{1,q}(\gamma)} \leq \gamma(B)^{1/q'} \|f\|_{L^q(T(B))}$$

(we equip the space  $T(B)$  with the product measure  $d\gamma(y)dt/t$ ). Hence  $\Lambda$  restricts to a bounded linear functional on  $L^q(T(B))$ , and is thus given by

$$\Lambda f = \iint_{T(B)} f \bar{g}_B d\gamma \frac{dt}{t}, \quad f \in L^q(T(B)),$$

for some  $g_B \in L^{q'}(T(B))$ , with the estimate

$$\|g_B\|_{L^{q'}(T(B))} \leq \gamma(B)^{1/q'} \|\Lambda\|_{\mathfrak{t}^{1,q'}(\gamma)^*}.$$

A single function  $g$  on  $D$  can then be obtained from the family  $(g_B)_{B \in \mathcal{B}_5}$  in a well-defined manner, since for any two balls  $B, B' \in \mathcal{B}_5$ , the functions  $g_B$  and  $g_{B'}$  agree on  $T(B) \cap T(B')$ . It remains to be checked that  $\|g\|_{\mathfrak{t}^{\infty,q'}(\gamma)} = \|\Lambda\|_{\mathfrak{t}^1(\gamma)^*}$ . On the one hand, for any  $B \in \mathcal{B}_5$  we have

$$\left( \iint_{T(B)} |g|^{q'} d\gamma \frac{dt}{t} \right)^{1/q'} = \|g_B\|_{L^{q'}(T(B))} \leq \gamma(B)^{1/q'} \|\Lambda\|_{\mathfrak{t}^{1,q'}(\gamma)^*}.$$

On the other hand, due to Theorem 2.4.5,  $\|\Lambda\|_{\mathfrak{t}^{1,q}(\gamma)^*}$  is achieved (up to a constant) by testing against all atoms, and so the proof is completed after checking that

$$\begin{aligned} |\Lambda a| &\leq \iint_{T(B)} |g\bar{a}| d\gamma \frac{dt}{t} \\ &\leq \left( \iint_{T(B)} |g|^{q'} d\gamma \frac{dt}{t} \right)^{1/q'} \left( \iint_{T(B)} |a|^q d\gamma \frac{dt}{t} \right)^{1/q} \\ &\leq \gamma(B)^{1/q'} \|g\|_{\mathfrak{t}^{\infty,q'}(X,\gamma)} \gamma(B)^{-1/q'} \\ &= \|g\|_{\mathfrak{t}^{\infty,q'}(\gamma)}, \end{aligned}$$

□

**Corollary 2.4.8.** *Suppose  $X$  is complete. For  $1 \leq p_0 \leq p_1 \leq \infty$  (excluding the case  $p_0 = p_1 = \infty$ ) and  $1 \leq q_0 \leq q_1 < \infty$ , we have  $[\mathfrak{t}^{p_0,q_0}(\gamma), \mathfrak{t}^{p_1,q_1}(\gamma)]_\theta = \mathfrak{t}^{p,q}(\gamma)$ , when  $1/p = (1-\theta)/p_0 + \theta/p_1$ ,  $1/q = (1-\theta)/q_0 + \theta/q_1$ , and  $0 \leq \theta \leq 1$ .*

*Proof.* This follows directly from Theorem 2.3.3 and Corollary 2.4.7, by convex reduction and reiteration (see Remark 2.3.4). □

**Corollary 2.4.9.** *Let  $q \geq 1$ . For all  $1 \leq p \leq q$  and  $\alpha \geq 1$  we have*

$$\|f\|_{\mathfrak{t}_\alpha^{p,q}(\gamma)} \lesssim C_{1,\alpha}^{1/q} C_{5,\alpha}^{1/p-1/q} \|f\|_{\mathfrak{t}^{p,q}(\gamma)}.$$

*Proof.* In order to argue by interpolation, consider first the case  $p = q$ :

$$\begin{aligned} \|f\|_{\mathfrak{t}_\alpha^{q,q}(\gamma)}^q &= \int_X \iint_{\Gamma^\alpha(x)} |f(y,t)|^q \frac{d\gamma(y)}{\gamma(B(y,t))} \frac{dt}{t} d\gamma(x) \\ &= \int_X \int_0^\infty |f(y,t)|^q \mathbf{1}_{(0,m(y))}(t) \frac{\gamma(B(y,\alpha t))}{\gamma(B(y,t))} \frac{dt}{t} d\gamma(y) \\ &\leq C_{1,\alpha} \iint_D |f(y,t)|^q d\gamma(y) \frac{dt}{t} \\ &= C_{1,\alpha} \|f\|_{\mathfrak{t}^{q,q}(\gamma)}^q. \end{aligned}$$

For  $p = 1$  we make use of the atomic decomposition. If  $a$  is a 5-atom associated with  $B \in \mathcal{B}_5$ , then, since  $\Gamma_\alpha(x) \cap T(B)$  is non-empty exactly when  $x \in \alpha B$ , we have

$$\begin{aligned} \|a\|_{\mathfrak{t}_\alpha^{1,q}(\gamma)} &\leq \gamma(\alpha B)^{1/q'} \|a\|_{\mathfrak{t}_\alpha^{q,q}(\gamma)} \\ &\leq C_{1,\alpha}^{1/q} \gamma(\alpha B)^{1/q'} \|a\|_{\mathfrak{t}^{q,q}(\gamma)} \\ &\leq C_{1,\alpha}^{1/q} \left( \frac{\gamma(\alpha B)}{\gamma(B)} \right)^{1/q'} \\ &\leq C_{1,\alpha}^{1/q} C_{5,\alpha}^{1-1/q}. \end{aligned}$$

Thus  $\|f\|_{\mathfrak{t}_\alpha^{1,q}(\gamma)} \leq C_{1,\alpha}^{1/q} C_{5,\alpha}^{1-1/q} \|f\|_{\mathfrak{t}^{1,q}(\gamma)}$  for all  $f \in \mathfrak{t}^{1,q}(\gamma)$ , and the result then follows by interpolation.  $\square$

*Remark 2.4.10.* Note that on  $(\mathbb{R}^n, dx)$  this gives the optimal dependence on  $\alpha$  for  $1 \leq p \leq 2$ , which we could not obtain from the vector-valued approach, since  $C_{1,\alpha}^{1/2} C_{5,\alpha}^{1/p-1/2} = \alpha^{n/p}$  (see Remark 2.3.4). On Gaussian  $\mathbb{R}^n$  this merely extends the aperture change to  $\mathfrak{t}^1(\gamma)$  with the constant  $e^{c\alpha^2}$ , the improvement from interpolation being immaterial.

## 2.5 Appendix 1: Local maximal functions

Here we present a brief justification of the boundedness of the maximal functions used above and in Appendix 2.6. We use dyadic methods, particularly the existence of finitely many ‘adjacent’ dyadic systems, combined with some methods from Martingale theory. At the end of this section we indicate another approach, which is more elementary but does not adapt well to vector-valued contexts.

By a *dyadic system* on a measure space  $(X, \gamma)$  we mean a countable collection  $\mathcal{D} = \{\mathcal{D}_k\}_{k \in \mathbb{Z}}$ , where each  $\mathcal{D}_k$  is a partition of  $X$  into measurable sets of finite nonzero measure, such that the containment relations

$$Q \in \mathcal{D}_k, \quad R \in \mathcal{D}_l, \quad l \geq k \implies R \subset Q \quad \text{or} \quad Q \cap R = \emptyset$$

hold. The elements of  $\mathcal{D}_k$  are called *dyadic cubes (of generation  $k$ )*.

Associated to each dyadic system  $\mathcal{D}$  is a *dyadic maximal function*, defined by

$$M_{\mathcal{D}}u(x) = \sup_{\substack{Q \ni x \\ Q \in \mathcal{D}}} \int_Q |u| d\gamma$$

for all  $u \in L^1_{\text{loc}}(\gamma)$ . Since  $M_{\mathcal{D}}$  coincides with the martingale maximal function for the (increasing) filtration  $(\mathcal{F}_k)_{k \in \mathbb{Z}}$  when each  $\mathcal{F}_k$  is the  $\sigma$ -algebra generated by  $\mathcal{D}_k$ , it follows that  $M_{\mathcal{D}}$  satisfies a weak type (1,1) inequality

$$\gamma(\{x \in X : M_{\mathcal{D}}u(x) > \lambda\}) \leq \frac{1}{\lambda} \|u\|_{L^1(\gamma)} \quad (2.9)$$

for all  $\lambda > 0$  (see for instance [91, Theorem 14.6] or [86, Chapter IV, Section 1]).

Now suppose that  $(X, d)$  is a geometrically doubling metric space. Hytönen and Kairema showed in [52] (see also [72]) the existence of a finite collection of *adjacent dyadic systems*.

**Theorem 2.5.1.** *There exists a finite collection  $\{\mathcal{D}_i\}_{i=1}^N$  of dyadic systems on  $X$ , with  $N$  bounded by a constant depending only on the geometric doubling constant of  $(X, d)$ , such that every open ball  $B \subset X$  is contained in a dyadic cube  $Q_B$  from one of the dyadic systems, with  $\text{diam}(Q_B) \leq c_X \text{diam}(B)$ .*

Now let  $(X, d, \mu)$ ,  $\gamma$ , and  $m$  be as in Section 2.2, and let  $\alpha > 0$ . Combining the theorem above with the weak type (1,1) estimate for the dyadic maximal function yields a corresponding weak type (1,1) estimate for the  $\alpha$ -local maximal operator  $M_\alpha$ .

Indeed, for each  $\alpha$ -admissible ball  $B \in \mathcal{B}_\alpha$  we have that  $B \subset Q_B$  for some dyadic cube  $Q_B$  that satisfies  $Q_B \subset c_X B$ , and so

$$\begin{aligned} \mathbf{1}_B(x) \int_B |u| d\gamma &\leq \mathbf{1}_{Q_B}(x) \frac{\gamma(Q_B)}{\gamma(B)} \int_{Q_B} |u| d\gamma \\ &\leq \mathbf{1}_{Q_B}(x) \frac{\gamma(c_X B)}{\gamma(B)} \int_{Q_B} |u| d\gamma \\ &\leq \mathbf{1}_{Q_B}(x) C_{\alpha, c_X} \int_{Q_B} |u| d\gamma. \end{aligned}$$

Here  $C_{\alpha, c_X}$  is the doubling constant from Remark 2.2.1. Summing over finitely many dyadic systems, we find that

$$M_\alpha u(x) \leq C_{\alpha, c_X} \sum_{\mathcal{D}} M_{\mathcal{D}} u(x),$$

and using the estimate (2.9) yields

$$\gamma(\{x \in X : M_\alpha u(x) > \lambda\}) \lesssim C_{\alpha, c_X} \|u\|_{L^1(\gamma)}$$

for all  $\lambda > 0$ .



Similarly, given a  $\sigma$ -finite measure space  $\Sigma$ , we can consider the  $\alpha$ -local maximal function  $\mathcal{M}_\alpha$ , given by

$$\mathcal{M}_\alpha U(x, s) = \sup_{\substack{B \in \mathcal{B}_\alpha \\ B \ni x}} \int_B |U(z, s)| d\gamma(z)$$

for  $U \in L^1_{\text{loc}}(\gamma; L^q(\Sigma))$  with  $q \in (1, \infty)$  (see [80] for a general overview). Again, this is controlled pointwise by a finite sum of its dyadic counterparts, that is,

$$\mathcal{M}_\alpha U(x, s) \leq C_{\alpha, c_X} \sum_{\mathcal{D}} \mathcal{M}_{\mathcal{D}} U(x, s) \quad (2.10)$$

for some finite collection of dyadic systems  $\mathcal{D}$ . The dyadic lattice maximal operators  $\mathcal{M}_{\mathcal{D}}$  are again amenable to Martingale theory. Indeed, according to the martingale version of Fefferman–Stein inequality (see [66, Subsection 3.1]) we have for  $1 < p < \infty$  that

$$\|\mathcal{M}_{\mathcal{D}} U\|_{L^p(\gamma; L^q(\Sigma))} \lesssim_{p,q} \|U\|_{L^p(\gamma; L^q(\Sigma))},$$

and consequently

$$\|\mathcal{M}_\alpha U\|_{L^p(\gamma; L^q(\Sigma))} \lesssim_{p,q} c_X C_{\alpha, c_X} \|U\|_{L^p(\gamma; L^q(\Sigma))}.$$

Although the explicit statement in [66] concerns the case of sequences, i.e. the case  $\Sigma = \mathbb{N}$ , it immediately extends to more general measure spaces  $\Sigma$  by means of *lattice finite representability*:  $L^q(\Sigma)$  is lattice finitely representable in  $\ell^q$  in the sense that for every finite dimensional sublattice  $E$  of  $L^q(\Sigma)$  and every  $\varepsilon > 0$  there exists a sublattice  $F$  of  $\ell^q$  and a lattice isomorphism  $\Phi: E \rightarrow F$  for which  $\|\Phi\| \|\Phi^{-1}\| \leq 1 + \varepsilon$  (see for instance [40] and the references therein). For boundedness of  $\mathcal{M}_{\mathcal{D}}$  it suffices to consider simple functions  $U: X \rightarrow L^q(\Sigma)$  and the boundedness is therefore transferable in lattice finite representability.

*Remark 2.5.2.* Martingale theory can be avoided by analysing  $M_\alpha$  by means of a ‘local Vitali covering lemma’, analogous to the usual analysis of the (global) maximal operator through the usual Vitali covering lemma. One can then prove the duality of  $\mathfrak{t}_\alpha^{p,q}$  and  $\mathfrak{t}_\alpha^{p',q'}$  for  $1 < p, q < \infty$ , and recover the boundedness of the projections  $N_\alpha$  by realising them as the adjoints of the (bounded) inclusions from  $\mathfrak{t}_\alpha^{p,q}$  into the appropriate  $L^q$ -valued  $L^p$ -space. This is the method of Bernal [23], used by the first author for global tent spaces in [3]. In this way we also avoid the use of the  $L^q(\Sigma)$ -valued maximal function  $\mathcal{M}_\alpha$ , but we do not achieve the potential generality of the above method.

## 2.6 Appendix 2: Cone covering lemma for non-negatively curved Riemannian manifolds

In this section we prove a stronger version of Lemma 2.4.4 that will be useful for the theory of vector-valued tent spaces. This is based on a ‘cone covering lemma’, the Euclidean version of which appears in [59, Lemma 10].

### 2.6.1 Review of non-negatively curved spaces

Recall that a complete length space  $(X, d)$  has *non-negative curvature* if and only if for every point  $x \in X$  and for every pair of geodesics  $\rho_1, \rho_2$  with  $\rho_1(0) = \rho_2(0) = x$ , the comparison angle

$$\angle_{\rho_1(t)x\rho_2(t)} := \cos^{-1} \left( \frac{d(x, \rho_1(t))^2 + d(x, \rho_2(t))^2 - d(\rho_1(t), \rho_2(t))^2}{2d(x, \rho_1(t))d(x, \rho_2(t))} \right)$$

is nonincreasing in  $t$  (this is the corresponding angle of a Euclidean triangle with sidelengths  $d(x, \rho_1(t))$ ,  $d(x, \rho_2(t))$ , and  $d(\rho_1(t), \rho_2(t))$ ). Actually, this monotonicity is a combination of the usual (local) definition of non-negative curvature and the conclusion of Topogonov’s theorem: see [27, Definition 4.3.1 and Theorem 10.3.1] for details.

We have the following simple corollary of this characterisation of non-negative curvature.

**Corollary 2.6.1.** *Suppose  $(X, d)$  is a complete length space with non-negative curvature. Let  $x, y, z \in X$ , let  $\rho_{xy}$  and  $\rho_{xz}$  be two unit speed minimising geodesics from  $x$  to  $y$  and  $z$  respectively, and denote the angle  $\angle(\rho'_{xy}(0), \rho'_{xz}(0))$  by  $\theta$ . Then*

$$d(y, z) \leq d(x, z) \tan \theta.$$

*Proof.* We have

$$\theta = \lim_{t \rightarrow 0} \angle(\rho'_{xy}(t), \rho'_{xz}(t)) \geq \theta'$$

by Topogonov’s theorem (as stated above), where  $\theta'$  is the comparison angle  $\tilde{\angle} yxz$ . By basic trigonometry,

$$\tan \theta' = \frac{d(y, z)}{d(x, z)},$$

and so we have

$$\tan \theta \geq \frac{d(y, z)}{d(x, z)}.$$

This yields the result. □

In particular, if  $\rho_1$  and  $\rho_2$  are two unit speed geodesics emanating from a point  $x \in X$  with  $\angle(\rho_1'(0), \rho_2'(0)) \leq \tan^{-1}(1/4)$ , then

$$d(\rho_1(t), \rho_2(t)) \leq t/4$$

for all  $t > 0$ , since  $d(\rho_2(0), \rho_2(t)) \leq t$ .

### 2.6.2 Cone covering

In this section, we assume that  $X$  is a complete geometrically doubling Riemannian manifold, so that  $(X, d)$  is a complete length space. We also fix  $\phi$  and  $m$  satisfying condition (A) as in Section 2.2 and assume in addition the following comparability condition:

(C) For every  $\alpha > 0$ , there exists a constant  $c_\alpha$  such that for all pairs of points  $x, y \in X$ ,

$$d(x, y) \leq \alpha m(x) \implies m(x) \leq c_\alpha m(y).$$

*Remark 2.6.2.* We could work in the context of complete geometrically doubling non-negatively curved length spaces; we have imposed smooth structure in order to use the language of tangent spaces rather than that of spaces of directions. The length space setting is only a small generalisation of the manifold setting, due to the fact that complete non-negatively curved length spaces are manifolds almost everywhere.

Given parameters  $\alpha \geq 1$  and  $\lambda \in (0, 1)$ , we define the extension of an open set  $E \subset X$  by

$$E_{\alpha, \lambda}^* := \bigcup \left\{ B \in \mathcal{B}_\alpha : \frac{\gamma(B \cap E)}{\gamma(B)} > \lambda \right\}.$$

Note that we can write

$$E_{\alpha, \lambda}^* = \{x \in X : M_\alpha \mathbf{1}_E(x) > \lambda\},$$

where  $M_\alpha$  is the  $\alpha$ -local maximal operator from Appendix 2.5, and so  $E_{\alpha, \lambda}^*$  is open. Furthermore, since for each  $\alpha \geq 1$  the local maximal function is of weak type  $(1, 1)$  with respect to  $\gamma$ , we have

$$\gamma(E_{\alpha, \lambda}^*) \leq \frac{C_\alpha}{\lambda} \gamma(E)$$

for all  $\lambda \in (0, 1)$ .

For all  $x \in X$ , for all unit tangent vectors  $v \in T_x X$  (recalling that we have assumed that  $X$  is a manifold), and for all  $t > 0$ , define the *sector*

$$R(v, t) := \bigcup_{0 \leq s \leq t} B(\rho(s), s/4)$$

opening from  $x$  in the direction of  $v$  along the unit speed geodesic  $\rho$  with  $\rho'(0) = v$ .

**Lemma 2.6.3.** *Let  $\beta \geq 1$ . There exists  $\alpha \geq 1$  and  $\lambda \in (0, 1)$  such that the following holds: if  $E \subset X$  is open and  $y \in R(v, t) \subset E$ , with  $v \in T_x X$  and  $0 < t \leq \beta m(x)$ , then  $B(y, 2t) \subset E_{\alpha, \lambda}^*$ .*

*Proof.* Suppose that  $E \subset X$  is open and  $y \in R(v, t) \subset E$ , with  $v \in T_x X$  and  $0 < t \leq \beta m(x)$ . We search for  $\alpha$  and  $\lambda$  so that

$$B(y, 2t) \in \mathcal{B}_\alpha \quad \text{and} \quad \frac{\gamma(B(y, 2t) \cap E)}{\gamma(B(y, 2t))} > \lambda.$$

Denote by  $\rho$  the unit speed geodesic determined by  $v$  and begin by observing that  $B(\rho(t), t/4) \subset R(v, t) \subset B(y, 2t) \cap E$ , while  $B(y, 2t) \subset B(\rho(t), 4t)$ , so that

$$\frac{\gamma(B(y, 2t) \cap E)}{\gamma(B(y, 2t))} \geq \frac{\gamma(B(\rho(t), t/4))}{\gamma(B(\rho(t), 4t))}.$$

Now  $d(x, \rho(t)) \leq t \leq \beta m(x)$ , and by (C) we have  $m(x) \leq c_\beta m(\rho(t))$ , so  $t \leq \beta m(x) \leq \beta c_\beta m(\rho(t))$ . This means that  $B(\rho(t), t/4)$  is  $\beta c_\beta/4$ -admissible, so that by (A),

$$\gamma(B(\rho(t), 4t)) \leq A_\beta \gamma\left(B\left(\rho(t), \frac{t}{4}\right)\right)$$

for some constant  $A_\beta$ . We may now choose  $\lambda < 1/A_\beta$  to get

$$\frac{\gamma(B(y, 2t) \cap E)}{\gamma(B(y, 2t))} > \lambda.$$

To choose  $\alpha$ , note that since  $d(x, y) \leq 2t \leq 2\beta m(x)$ , we have  $m(x) \leq c_{2\beta} m(y)$ , and so  $t \leq \beta c_{2\beta} m(y)$ . In order to have  $B(y, 2t) \in \mathcal{B}_\alpha$ , we choose  $\alpha = 2\beta c_{2\beta}$ . By the definition of the extension, we now have  $B(y, 2t) \subset E_{\alpha, \lambda}^*$ .  $\square$

Dictated by the final paragraph in the proof of the following lemma, we now fix  $\beta = c_1$ , and choose  $\alpha$  and  $\lambda$  in accordance with Lemma 2.6.3. We also write  $E^* = E_{\alpha, \lambda}^*$ . Recall that the admissible tent  $T(O)$  over an open set  $O \subset X$  is given by

$$T(O) := D \setminus \Gamma(O^c),$$

where  $\Gamma(O^c) := \bigcup_{x \in O^c} \Gamma(x)$ .

**Lemma 2.6.4** (Cone covering lemma). *Assume that  $X$  is non-negatively curved, and let  $E \subset X$  be a bounded open set. Then for every  $x \in E$  there exist finitely many points  $x_1, \dots, x_N \in X \setminus E$ , with  $N$  depending only on the dimension of  $X$ , such that*

$$\Gamma(x) \setminus T(E^*) \subset \bigcup_{m=1}^N \Gamma(x_m).$$

*Proof.* Let  $x \in E$  and pick unit vectors  $v_1, \dots, v_N \in T_x X$  so that every  $v \in T_x X$  has  $\angle(v, v_m) \leq \tan^{-1}(1/4)$  for some  $m = 1, \dots, N$ . For each  $m$ , denote by  $\rho_m$  the unit speed geodesic determined by  $v_m$ , and let  $t_m > 0$  be the minimal number ( $E$  is bounded) for which  $\overline{B}(\rho_m(t_m), t_m/4)$  intersects  $X \setminus E$ , so that we may choose an  $x_m \in (X \setminus E) \cap \overline{B}(\rho_m(t_m), t_m/4)$ . Note that now  $R(v_m, t_m) \subset E$  for each  $m$ .

Letting  $(y, t) \in \Gamma(x) \setminus T(E^*)$ , we need to show that  $d(y, x_m) < t$  for some  $m$ . By completeness of  $X$ , we may choose a unit speed minimising geodesic  $\rho$  from  $x$  to  $y$  and then fix an  $m$  so that  $\angle(\rho'(0), v_m) \leq \tan^{-1}(1/4)$ . Corollary 2.6.1 guarantees that  $y \in R(v_m, d(x, y))$ .

Suppose first that  $x$  is close to  $E^c$  in the direction of  $v_m$ , in the sense that  $t_m \leq \beta m(x)$ . If  $d(x, y) > t_m$ , then by Corollary 2.6.1  $\rho(t_m)$  is in  $\overline{B}(\rho_m(t_m), t_m/4)$ , and so

$$\begin{aligned} d(y, x_m) &\leq d(y, \rho(t_m)) + d(\rho(t_m), x_m) \\ &\leq d(y, \rho(t_m)) + \frac{t_m}{2} \\ &\leq d(y, \rho(t_m)) + d(\rho(t_m), x) \\ &= d(y, x) < t. \end{aligned}$$

On the other hand, if  $d(x, y) \leq t_m$ , then  $y \in R(v_m, t_m)$ —that is,  $y \in B(\rho_m(s), s/4)$  for some  $0 \leq s \leq t_m$ —and so

$$\begin{aligned} d(y, x_m) &\leq d(y, \rho_m(s)) + d(\rho_m(s), \rho_m(t_m)) + d(\rho_m(t_m), x_m) \\ &\leq \frac{s}{4} + t_m - s + \frac{t_m}{4} \leq 2t_m. \end{aligned}$$

According to Lemma 2.6.3,  $B(y, 2t_m) \subset E^*$ , but since  $(y, t) \notin T(E^*)$  implies that  $B(y, t) \not\subset E^*$ , we must have  $2t_m < t$ .

Second, we show that it is not possible to have  $t_m > \beta m(x)$  with  $\beta = c_1$ . Note first that since  $d(x, y) < t < m(y)$ , we have by (C) that  $t < m(y) \leq c_1 m(x)$ . If indeed we had  $t_m > c_1 m(x)$ , then  $y \in R(v_m, c_1 m(x)) \subset R(v_m, t_m) \subset E$ . Invoking Lemma 2.6.3 gives  $B(y, c_1 m(x)) \subset B(y, 2c_1 m(x)) \subset E^*$ , while  $B(y, t) \not\subset E^*$  and so  $c_1 m(x) < t$ , which is a contradiction.  $\square$

The cone covering lemma allows stronger pointwise estimation of the functional  $\mathcal{A}_q$  when  $q \geq 1$  (cf. Lemma 2.4.4):

**Corollary 2.6.5.** *Assume that  $X$  is non-negatively curved. Suppose  $1 \leq q < \infty$ , and let  $f$  be a function on  $D$  with bounded support. Let  $\lambda > 0$  and write  $E = \{x \in X : \mathcal{A}_q f(x) > \lambda\}$ . Then*

$$\mathcal{A}_q(f1_{D \setminus T(E^*)})(x) \lesssim_{\dim X} \lambda \quad \text{for all } x \in X.$$

*Proof.* If  $x \in X \setminus E$ , then

$$\mathcal{A}_q(f1_{D \setminus T(E^*)})(x) \leq \mathcal{A}_q f(x) \leq \lambda$$

by the definition of  $E$ . So let  $x \in E$ . Since  $E$  is a bounded open set, we may use Lemma 2.6.4 to pick  $x_1, \dots, x_N \in X \setminus E$  (with  $N$  depending only on the dimension of  $X$ ) such that

$$\Gamma(x) \setminus T(E^*) \subset \bigcup_{m=1}^N \Gamma(x_m).$$

We can then estimate

$$\begin{aligned} \mathcal{A}_q(f1_{D \setminus T(E^*)})(x) &= \left( \iint_{\Gamma(x) \setminus T(E^*)} |f(y, t)|^q \frac{d\gamma(y)}{\gamma(B(y, t))} \frac{dt}{t} \right)^{1/q} \\ &\leq \sum_{m=1}^N \left( \iint_{\Gamma(x_m)} |f(y, t)|^q \frac{d\gamma(y)}{\gamma(B(y, t))} \frac{dt}{t} \right)^{1/q} \leq N\lambda, \end{aligned}$$

proving the corollary. □

*Remark 2.6.6.* At the time of writing we do not know of any doubling Riemannian manifolds (equipped with  $\phi$  and  $m$ ) for which the cone covering lemma fails. It would be interesting to determine more precisely which spaces admit cone coverings of the type above.

# Chapter 3

## Interpolation and embeddings of weighted tent spaces

### Abstract

Given a metric measure space  $X$ , we consider a scale of function spaces  $T_s^{p,q}(X)$ , called the *weighted tent space scale*. This is an extension of the tent space scale of Coifman, Meyer, and Stein. Under various geometric assumptions on  $X$  we identify some associated interpolation spaces, in particular certain real interpolation spaces within the reflexive range. These are identified with a new scale of function spaces, which we call *Z-spaces*, that have recently appeared in the work of Barton and Mayboroda on elliptic boundary value problems with boundary data in Besov spaces. We also prove Hardy–Littlewood–Sobolev-type embeddings between weighted tent spaces.

### 3.1 Introduction

The *tent spaces*, denoted  $T^{p,q}$ , are a scale of function spaces first introduced by Coifman, Meyer, and Stein [32, 33] which have had many applications in harmonic analysis and partial differential equations. In some of these applications ‘weighted’ tent spaces have been used implicitly. These spaces, which we denote by  $T_s^{p,q}$ , seem not to have been considered as forming a scale of function spaces in their own right until the work of Hofmann, Mayboroda, and McIntosh [50, §8.3], in which factorisation and complex interpolation theorems are obtained for them.

In this article we further explore the weighted tent space scale. In the interests of generality, we consider weighted tent spaces  $T_s^{p,q}(X)$  associated with a

metric measure space  $X$ , although our theorems are new even in the classical case where  $X = \mathbb{R}^n$  equipped with the Lebesgue measure. Under sufficient geometric assumptions on  $X$  (ranging from the doubling condition to the assumption that  $X = \mathbb{R}^n$ ), we uncover two previously unknown novelties of the weighted tent space scale.

First, we identify some real interpolation spaces between  $T_{s_0}^{p_0,q}$  and  $T_{s_1}^{p_1,q}$  whenever  $s_0 \neq s_1$ . In Theorem 3.3.4 we prove that

$$(T_{s_0}^{p_0,q}, T_{s_1}^{p_1,q})_{\theta, p_\theta} = Z_{s_\theta}^{p_\theta,q} \quad (3.1)$$

for appropriately defined parameters, where the scale of ‘ $Z$ -spaces’ is defined in Definition 3.3.3. We require  $p_0, p_1, q > 1$  in this identification, but in Theorem 3.3.9 we show that in the Euclidean setting the result holds for all  $p_0, p_1 > 0$  and  $q \geq 1$ . In the Euclidean setting,  $Z$ -spaces have appeared previously in the work of Barton and Mayboroda [21]. In their notation we have  $Z_s^{p,q}(\mathbb{R}^n) = L(p, ns+1, q)$ . Barton and Mayboroda show that these function spaces are useful in the study of elliptic boundary value problems with boundary data in Besov spaces. The connection with weighted tent spaces shown here is new.

Second, we have continuous embeddings

$$T_{s_0}^{p_0,q} \hookrightarrow T_{s_1}^{p_1,q}$$

whenever the parameters satisfy the relation

$$s_1 - s_0 = \frac{1}{p_1} - \frac{1}{p_0}. \quad (3.2)$$

This is Theorem 3.3.19. Thus a kind of Hardy–Littlewood–Sobolev embedding theorem holds for the weighted tent space scale, and by analogy we are justified in referring to the parameter  $s$  in  $T_s^{p,q}$  as a *regularity* parameter.

We also identify complex interpolation spaces between weighted tent spaces in the Banach range. This result is already well-known in the Euclidean setting, and its proof does not involve any fundamentally new arguments, but we include it here for completeness.

These results in this paper will play a crucial role in forthcoming work,<sup>1</sup> in which we will use weighted tent spaces and  $Z$ -spaces to construct abstract homogeneous Hardy–Sobolev and Besov spaces associated with elliptic differential operators with rough coefficients. This will be an extension of the abstract Hardy space techniques initiated independently by Auscher, McIntosh, and Russ [13] and Hofmann and Mayboroda [49].

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<sup>1</sup>See Part II of this thesis.



## Notation

Given a measure space  $(X, \mu)$ , we write  $L^0(X)$  for the set of  $\mu$ -measurable functions with values in the extended complex numbers  $\mathbb{C} \cup \{\pm\infty, \pm i\infty\}$ . As usual, by a ‘measurable function’, we actually mean an equivalence class of measurable functions which are equal except possibly on a set of measure zero. We will say that a function  $f \in L^0(X)$  is essentially supported in a subset  $E \subset X$  if we have  $\mu\{x \in X \setminus E : f(x) \neq 0\} = 0$ .

A *quasi-Banach space* is a complete quasi-normed vector space; see for example [56, §2] for further information. If  $B$  is a quasi-Banach space, we will write the quasi-norm of  $B$  as either  $\|\cdot\|_B$  or  $\|\cdot\|_B$ , according to typographical needs.

For  $1 \leq p \leq \infty$ , we let  $p'$  denote the Hölder conjugate of  $p$ , which is defined by the relation

$$1 = \frac{1}{p} + \frac{1}{p'},$$

with  $1/\infty := 0$ . For  $0 < p, q \leq \infty$ , we define the number

$$\delta_{p,q} := \frac{1}{q} - \frac{1}{p},$$

again with  $1/\infty := 0$ . This shorthand will be used often throughout this article. We will frequently use the identities

$$\begin{aligned} \delta_{p,q} + \delta_{q,r} &= \delta_{p,r}, \\ \delta_{p,q} &= \delta_{q',p'}, \\ 1/q &= \delta_{\infty,q} = \delta_{q',1}. \end{aligned}$$

As is now standard in harmonic analysis, we write  $a \lesssim b$  to mean that  $a \leq Cb$  for some unimportant constant  $C \geq 1$  which will generally change from line to line. We also write  $a \lesssim_{c_1, c_2, \dots} b$  to mean that  $a \leq C(c_1, c_2, \dots)b$ .

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## 3.2 Preliminaries

### 3.2.1 Metric measure spaces

A *metric measure space* is a triple  $(X, d, \mu)$ , where  $(X, d)$  is a nonempty metric space and  $\mu$  is a Borel measure on  $X$ . For every  $x \in X$  and  $r > 0$ , we write  $B(x, r) := \{y \in X : d(x, y) < r\}$  for the ball of radius  $r$ , and we also write  $V(x, r) := \mu(B(x, r))$  for the volume of this ball. The *generalised half-space* associated with  $X$  is the set  $X^+ := X \times \mathbb{R}_+$ , equipped with the product topology and the product measure  $d\mu(y) dt/t$ .

We say that  $(X, d, \mu)$  is *nondegenerate* if

$$0 < V(x, r) < \infty \quad \text{for all } x \in X \text{ and } r > 0. \quad (3.3)$$

This immediately implies that the measure space  $(X, \mu)$  is  $\sigma$ -finite, as  $X$  may be written as an increasing sequence of balls

$$X = \bigcup_{n \in \mathbb{N}} B(x_0, n) \quad (3.4)$$

for any point  $x_0 \in X$ . Nondegeneracy also implies that the metric space  $(X, d)$  is separable [24, Proposition 1.6]. To rule out pathological behaviour (which is not particularly interesting from the viewpoint of tent spaces), we will always assume nondegeneracy.

Generally we will need to make further geometric assumptions on  $(X, d, \mu)$ . In this article, the following two conditions will be used at various points. We say that  $(X, d, \mu)$  is *doubling* if there exists a constant  $C \geq 1$  such that

$$V(x, 2r) \leq CV(x, r) \quad \text{for all } (x, r) \in X^+.$$

A consequence of the doubling condition is that there exists a minimal number  $n \geq 0$ , called the *doubling dimension* of  $X$ , and a constant  $C \geq 1$  such that

$$V(x, R) \leq C(R/r)^n V(x, r)$$

for all  $x \in X$  and  $0 < r \leq R < \infty$ .

For  $n > 0$ , we say that  $(X, d, \mu)$  is *AD-regular of dimension  $n$*  if there exists a constant  $C \geq 1$  such that

$$C^{-1}r^n \leq V(x, r) \leq Cr^n \quad (3.5)$$

for all  $x \in X$  and all  $r < \text{diam}(X)$ . One can show that AD-regularity (of some dimension) implies doubling. Note that if  $X$  is unbounded and AD-regular of dimension  $n$ , then (3.5) holds for all  $x \in X$  and all  $r > 0$ .

### 3.2.2 Unweighted tent spaces

Throughout this section we suppose that  $(X, d, \mu)$  is a nondegenerate metric measure space. We will not assume any further geometric conditions on  $X$  without explicit mention. All of the results here are known, at least in some form. We provide statements for ease of reference and some proofs for completeness.

For  $x \in X$  we define the *cone with vertex  $x$*  by

$$\Gamma(x) := \{(y, t) \in X^+ : y \in B(x, t)\},$$

and for each ball  $B \subset X$  we define the *tent with base  $B$*  by

$$T(B) := X^+ \setminus \left( \bigcup_{x \notin B} \Gamma(x) \right).$$

Equivalently,  $T(B)$  is the set of points  $(y, t) \in X^+$  such that  $B(y, t) \subset B$ . From this characterisation it is clear that if  $(y, t) \in T(B)$ , then  $t \leq r_B$ , where we define

$$r_B := \sup\{r > 0 : B(y, r) \subset B \text{ for some } y \in X\}.$$

Note that it is possible to have  $r_{B(y, t)} > t$ .

Fix  $q \in (0, \infty)$  and  $\alpha \in \mathbb{R}$ . For  $f \in L^0(X^+)$ , define functions  $\mathcal{A}^q f$  and  $\mathcal{C}_\alpha^q f$  on  $X$  by

$$\mathcal{A}^q f(x) := \left( \iint_{\Gamma(x)} |f(y, t)|^q \frac{d\mu(y)}{V(y, t)} \frac{dt}{t} \right)^{1/q} \quad (3.6)$$

and

$$\mathcal{C}_\alpha^q f(x) := \sup_{B \ni x} \frac{1}{\mu(B)^\alpha} \left( \frac{1}{\mu(B)} \iint_{T(B)} |f(y, t)|^q d\mu(y) \frac{dt}{t} \right)^{1/q} \quad (3.7)$$

for all  $x \in X$ , where the supremum in (3.7) is taken over all balls  $B \subset X$  containing  $x$ . We abbreviate  $\mathcal{C}^q := \mathcal{C}_0^q$ . Note that the integrals above are always

defined (though possibly infinite) as the integrands are non-negative, and so we need not assume any local  $q$ -integrability of  $f$ . We also define

$$\mathcal{A}^\infty f(x) := \operatorname{ess\,sup}_{(y,t) \in \Gamma(x)} |f(y,t)| \quad (3.8)$$

and

$$\mathcal{C}_\alpha^\infty f(x) := \sup_{B \ni x} \frac{1}{\mu(B)^{1+\alpha}} \operatorname{ess\,sup}_{(y,t) \in T(B)} |f(y,t)|.$$

**Lemma 3.2.1.** *Suppose that  $q \in (0, \infty]$ ,  $\alpha \in \mathbb{R}$ , and  $f \in L^0(X^+)$ . Then the functions  $\mathcal{A}^q f$  and  $\mathcal{C}_\alpha^q f$  are lower semicontinuous.*

*Proof.* For  $q \neq \infty$  see [3, Lemmas A.6 and A.7]. It remains only to show that  $\mathcal{A}^\infty f$  and  $\mathcal{C}_\alpha^\infty f$  are lower semicontinuous for  $f \in L^0(X^+)$ .

For each  $s > 0$  write

$$\Gamma(x) + s := \{(y,t) \in X^+ : (y,t-s) \in \Gamma(x)\} = \{(y,t) \in X^+ : y \in B(x,t-s)\}.$$

Geometrically  $\Gamma(x) + s$  is a ‘vertically translated’ cone, and  $\Gamma(x) + s \supset \Gamma(x) + r$  for all  $r < s$ . The triangle inequality implies that

$$\Gamma(x) + s \subset \Gamma(x') \quad \text{for all } x' \in B(x,s).$$

To show that  $\mathcal{A}^\infty f$  is lower semicontinuous, suppose that  $x \in X$  and  $\lambda > 0$  are such that  $(\mathcal{A}^\infty f)(x) > \lambda$ . Then the set  $O := \{(y,t) \in \Gamma(x) : |f(y,t)| > \lambda\}$  has positive measure. We have

$$O = \bigcup_{n=1}^{\infty} O \cap (\Gamma(x) + n^{-1}).$$

Since the sequence of sets  $O \cap (\Gamma(x) + n^{-1})$  is increasing in  $n$ , and since  $O$  has positive measure, we find that there exists  $n \in \mathbb{N}$  such that  $O \cap (\Gamma(x) + n^{-1})$  has positive measure. Thus for all  $x' \in B(x, n^{-1})$ ,

$$\{(y,t) \in \Gamma(x') : |f(y,t)| > \lambda\} \supset O \cap (\Gamma(x) + n^{-1})$$

has positive measure, and so  $(\mathcal{A}^\infty f)(x') > \lambda$ . Therefore  $\mathcal{A}^\infty f$  is lower semicontinuous.

The argument for  $\mathcal{C}_\alpha^\infty$  is simpler. We have  $(\mathcal{C}_\alpha^\infty f)(x) > \lambda$  if and only if there exists a ball  $B \ni x$  such that

$$\frac{1}{\mu(B)^{1+\alpha}} \operatorname{ess\,sup}_{(y,t) \in T(B)} |f(y,t)| > \lambda.$$

This immediately yields  $(\mathcal{C}_\alpha^\infty f)(x') > \lambda$  for all  $x' \in B$ , and so  $\mathcal{C}_\alpha^\infty f$  is lower semicontinuous.  $\square$

**Definition 3.2.2.** For  $p \in (0, \infty)$  and  $q \in (0, \infty]$ , the *tent space*  $T^{p,q}(X)$  is the set

$$T^{p,q}(X) := \{f \in L^0(X^+) : \mathcal{A}^q f \in L^p(X)\}$$

equipped with the quasi-norm

$$\|f\|_{T^{p,q}(X)} := \|\mathcal{A}^q f\|_{L^p(X)}.$$

We define  $T^{\infty,q}(X)$  by

$$T^{\infty,q}(X) := \{f \in L^0(X^+) : \mathcal{C}^q f \in L^\infty(X)\}$$

equipped with the corresponding quasi-norm. We define  $T^{\infty,\infty}(X) := L^\infty(X^+)$  with equal norms.

For the sake of notational clarity, we will write  $T^{p,q}$  rather than  $T^{p,q}(X)$  unless we wish to emphasise a particular choice of  $X$ . Although we will always refer to tent space ‘quasi-norms’, these are norms when  $p, q \geq 1$ .

*Remark 3.2.3.* Our definition of  $\mathcal{A}^\infty f$  gives a function which is less than or equal to the corresponding function defined by Coifman, Meyer, and Stein [33], which uses suprema instead of essential suprema. We also do not impose any continuity conditions in our definition of  $T^{p,\infty}$ . Therefore our space  $T^{p,\infty}(\mathbb{R}^n)$  is strictly larger than the Coifman–Meyer–Stein version.

By a *cylinder* we mean a subset  $C \subset X^+$  of the form  $C = B(x, r) \times (a, b)$  for some  $(x, r) \in X^+$  and  $0 < a < b < \infty$ . We say that a function  $f \in L^0(X^+)$  is *cylindrically supported* if it is essentially supported in a cylinder. In general cylinders may not be precompact, and so the notion of cylindrical support is more general than that of compact support. For all  $p, q \in (0, \infty]$  we define

$$T^{p,q;c} := \{f \in T^{p,q} : f \text{ is cylindrically supported}\}.$$

and

$$L_c^p(X^+) := \{f \in L^p(X^+) : f \text{ is cylindrically supported}\}.$$

A straightforward application of the Fubini–Tonelli theorem shows that for all  $q \in (0, \infty)$  and for all  $f \in L^0(X^+)$ ,

$$\|f\|_{T^{q,q}} = \|f\|_{L^q(X^+)},$$

and so  $T^{q,q} = L^q(X^+)$ . When  $q = \infty$  this is true by definition.

**Proposition 3.2.4.** *For all  $p, q \in (0, \infty)$ , the subspace  $T^{p,q;c} \subset T^{p,q}$  is dense in  $T^{p,q}$ . Furthermore, if  $X$  is doubling, then for all  $p, q \in (0, \infty]$ ,  $T^{p,q}$  is complete, and when  $p, q \neq \infty$ ,  $L_c^q(X^+)$  is densely contained in  $T^{p,q}$ .*

*Proof.* The second statement has already been proven in [3, Proposition 3.5],<sup>2</sup> so we need only prove the first statement. Suppose  $f \in T^{p,q}$  and fix a point  $x_0 \in X$ . For each  $k \in \mathbb{N}$ , define

$$C_k := B(x_0, k) \times (k^{-1}, k) \quad \text{and} \quad f_k := \mathbf{1}_{C_k} f.$$

Then each  $f_k$  is cylindrically supported. We have

$$\begin{aligned} \lim_{k \rightarrow \infty} \|f - f_k\|_{T^{p,q}}^p &= \lim_{k \rightarrow \infty} \int_X \mathcal{A}^q(\mathbf{1}_{C_k^c} f)(x)^p d\mu(x) \\ &= \int_X \lim_{k \rightarrow \infty} \mathcal{A}^q(\mathbf{1}_{C_k^c} f)(x)^p d\mu(x) \\ &= \int_X \left( \lim_{k \rightarrow \infty} \iint_{\Gamma(x)} |(\mathbf{1}_{C_k^c} f)(y, t)|^q \frac{d\mu(y)}{V(y, t)} \frac{dt}{t} \right)^{p/q} d\mu(x) \\ &= \int_X \left( \iint_{\Gamma(x)} \lim_{k \rightarrow \infty} |(\mathbf{1}_{C_k^c} f)(y, t)|^q \frac{d\mu(y)}{V(y, t)} \frac{dt}{t} \right)^{p/q} d\mu(x) \\ &= 0. \end{aligned}$$

All interchanges of limits and integrals follow from monotone convergence. Hence we have  $f = \lim_{k \rightarrow \infty} f_k$ , which completes the proof.  $\square$

Recall the following duality from [3, Proposition 3.10].

**Proposition 3.2.5.** *Suppose that  $X$  is doubling,  $p \in [1, \infty)$ , and  $q \in (1, \infty)$ . Then the  $L^2(X^+)$  inner product*

$$\langle f, g \rangle := \iint_{X^+} f(x, t) \overline{g(x, t)} d\mu(x) \frac{dt}{t} \quad (3.9)$$

*identifies the dual of  $T^{p,q}$  with  $T^{p',q'}$ .*

Suppose that  $p \in (0, 1]$ ,  $q \in [p, \infty]$ , and  $B \subset X$  is a ball. We say that a function  $a \in L^0(X^+)$  is a  $T^{p,q}$  atom (associated with  $B$ ) if  $a$  is essentially supported in  $T(B)$  and if the size estimate

$$\|a\|_{T^{p,q}} \leq \mu(B)^{\delta_{p,q}}$$

holds (recall that  $\delta_{p,q} := q^{-1} - p^{-1}$ ). A short argument shows that if  $a$  is a  $T^{p,q}$ -atom, then  $\|a\|_{T^{p,q}} \leq 1$ .

---

<sup>2</sup>The cases where  $q = \infty$  are not covered there. The same proof works—the only missing ingredient is Lemma 3.4.1, which we defer to the end of the article.

**Theorem 3.2.6** (Atomic decomposition). *Suppose that  $X$  is doubling. Let  $p \in (0, 1]$  and  $q \in [p, \infty]$ . Then a function  $f \in L^0(X^+)$  is in  $T^{p,q}$  if and only if there exists a sequence  $(a_k)_{k \in \mathbb{N}}$  of  $T^{p,q}$ -atoms and a sequence  $(\lambda_k)_{k \in \mathbb{N}} \in \ell^p(\mathbb{N})$  such that*

$$f = \sum_{k \in \mathbb{N}} \lambda_k a_k \quad (3.10)$$

*with convergence in  $T^{p,q}$ . Furthermore, we have*

$$\|f\|_{T^{p,q}} \simeq \inf \|\lambda_k\|_{\ell^p(\mathbb{N})},$$

*where the infimum is taken over all decompositions of the form (3.10).*

This is proven by Russ when  $q = 2$  [81], and the same proof works for general  $q \in [p, \infty)$ . For  $q = \infty$  we need to combine the original argument of Coifman, Meyer, and Stein [33, Proposition 2] with that of Russ. We defer this to Section 3.4.2.

### 3.2.3 Weighted tent spaces: definitions, duality, and atoms

We continue to suppose that  $(X, d, \mu)$  is a nondegenerate metric measure space, and again we make no further assumptions without explicit mention.

For each  $s \in \mathbb{R}$ , we can define an operator  $V^s$  on  $L^0(X^+)$  by

$$(V^s f)(x, t) := V(x, t)^s f(x, t)$$

for all  $(x, t) \in X^+$ . Note that for  $r, s \in \mathbb{R}$  the equality  $V^r V^s = V^{r+s}$  holds, and also that  $V^0$  is the identity operator. Using these operators we define modified tent spaces, which we call *weighted tent spaces*, as follows.

**Definition 3.2.7.** For  $p \in (0, \infty)$ ,  $q \in (0, \infty]$ , and  $s \in \mathbb{R}$ , the *weighted tent space*  $T_s^{p,q}$  is the set

$$T_s^{p,q} := \{f \in L^0(X^+) : V^{-s} f \in T^{p,q}\}$$

equipped with the quasi-norm

$$\|f\|_{T_s^{p,q}} := \|V^{-s} f\|_{T^{p,q}}.$$

We also define  $T_s^{\infty, \infty}$  in this way. For  $q \neq \infty$ , and with an additional parameter  $\alpha \in \mathbb{R}$ , we define  $T_{s;\alpha}^{\infty, q}$  by the quasi-norm

$$\|f\|_{T_{s;\alpha}^{\infty, q}} := \left\| \mathcal{C}_\alpha^q(V^{-s} f) \right\|_{L^\infty(X)}.$$

Note that  $T_{0;0}^{\infty, q} = T^{\infty, q}$ . We write  $T_s^{\infty, q} := T_{s;0}^{\infty, q}$ .

*Remark 3.2.8.* The weighted tent space quasi-norms of Hofmann, Mayboroda, and McIntosh [50, §8.3] (with  $p \neq \infty$ ) and Huang [51] (including  $p = \infty$  with  $\alpha = 0$ ) are given by

$$\|f\|_{T_s^{p,q}(\mathbb{R}^n)} := \left\| (y, t) \mapsto t^{-s} f(y, t) \right\|_{T^{p,q}(\mathbb{R}^n)}, \quad (3.11)$$

which are equivalent to those of our spaces  $T_{s/n}^{p,q}(\mathbb{R}^n)$ . In general, when  $X$  is unbounded and AD-regular of dimension  $n$ , the quasi-norm in (3.11) (with  $X$  replacing  $\mathbb{R}^n$ ) is equivalent to that of our  $T_{s/n}^{p,q}$ . We have chosen the convention of weighting with ball volumes, rather than with the variable  $t$ , because this leads to more geometrically intrinsic function spaces and supports embedding theorems under weaker assumptions.

For all  $r, s \in \mathbb{R}$ , the operator  $V^r$  is an isometry from  $T_s^{p,q}$  to  $T_{s+r}^{p,q}$ . The operator  $V^{-r}$  is also an isometry, now from  $T_{s+r}^{p,q}$  to  $T_s^{p,q}$ , and so for fixed  $p$  and  $q$  the weighted tent spaces  $T_s^{p,q}$  are isometrically isomorphic for all  $s \in \mathbb{R}$ . Thus by Proposition 3.2.4, when  $X$  is doubling, the spaces  $T_s^{p,q}$  are all complete.

Recall the  $L^2(X^+)$  inner product (3.9), which induces a duality pairing between  $T^{p,q}$  and  $T^{p',q'}$  for appropriate  $p$  and  $q$  when  $X$  is doubling. For all  $s \in \mathbb{R}$  and all  $f, g \in L^2(X^+)$  we have the equality

$$\langle f, g \rangle = \langle V^{-s} f, V^s g \rangle, \quad (3.12)$$

which yields the following duality result.

**Proposition 3.2.9.** *Suppose that  $X$  is doubling,  $p \in [1, \infty)$ ,  $q \in (1, \infty)$ , and  $s \in \mathbb{R}$ . Then the  $L^2(X^+)$  inner product (3.9) identifies the dual of  $T_s^{p,q}$  with  $T_{-s}^{p',q'}$ .*

*Proof.* If  $f \in T_s^{p,q}$  and  $g \in T_{-s}^{p',q'}$ , then we have  $V^{-s} f \in T^{p,q}$  and  $V^s g \in T^{p',q'}$ , so by Proposition 3.2.5 and (3.12) we have

$$|\langle f, g \rangle| \lesssim \|V^{-s} f\|_{T^{p,q}} \|V^s g\|_{T^{p',q'}} = \|f\|_{T_s^{p,q}} \|g\|_{T_{-s}^{p',q'}}.$$

Conversely, if  $\varphi \in (T_s^{p,q})'$ , then the map  $\tilde{f} \mapsto \varphi(V^s \tilde{f})$  determines a bounded linear functional on  $T^{p,q}$  with norm dominated by  $\|\varphi\|$ . Hence by Proposition 3.2.5 there exists a function  $\tilde{g} \in T^{p',q'}$  with  $\|\tilde{g}\|_{T^{p',q'}} \lesssim \|\varphi\|$  such that

$$\varphi(f) = \varphi(V^s(V^{-s} f)) = \langle V^{-s} f, \tilde{g} \rangle = \langle f, V^{-s} \tilde{g} \rangle$$

for all  $f \in T_s^{p,q}$ . Since

$$\|V^{-s} \tilde{g}\|_{T_{-s}^{p',q'}} = \|\tilde{g}\|_{T^{p',q'}} \lesssim \|\varphi\|,$$

we are done.  $\square$



There is also a duality result for  $p < 1$  which incorporates the spaces  $T_{s;\alpha}^{\infty,q}$  with  $\alpha > 0$ . Before we can prove it, we need to discuss atomic decompositions.

Suppose that  $p \in (0, 1]$ ,  $q \in [p, \infty]$ ,  $s \in \mathbb{R}$ , and  $B \subset X$  is a ball. We say that a function  $a \in L^0(X^+)$  is a  $T_s^{p,q}$ -atom (associated with  $B$ ) if  $V^{-s}a$  is a  $T^{p,q}$ -atom. This is equivalent to demanding that  $a$  is essentially supported in  $T(B)$  and that

$$\|a\|_{T_s^{p,q}} \leq \mu(B)^{\delta_{p,q}}.$$

The atomic decomposition theorem for unweighted tent spaces (Theorem 3.2.6) immediately implies its weighted counterpart.

**Proposition 3.2.10** (Atomic decomposition for weighted tent spaces). *Suppose that  $X$  is doubling. Let  $p \in (0, 1]$ ,  $q \in [p, \infty]$ , and  $s \in \mathbb{R}$ . Then a function  $f \in L^0(X^+)$  is in  $T_s^{p,q}$  if and only if there exists a sequence  $(a_k)_{k \in \mathbb{N}}$  of  $T_s^{p,q}$ -atoms and a sequence  $(\lambda_k)_{k \in \mathbb{N}} \in \ell^p(\mathbb{N})$  such that*

$$f = \sum_{k \in \mathbb{N}} \lambda_k a_k \quad (3.13)$$

with convergence in  $T_s^{p,q}$ . Furthermore, we have

$$\|f\|_{T_s^{p,q}} \simeq \inf \|\lambda_k\|_{\ell^p(\mathbb{N})},$$

where the infimum is taken over all decompositions of the form (3.13).

Using this, we can prove the following duality result for  $p < 1$ .

**Theorem 3.2.11.** *Suppose that  $X$  is doubling,  $p \in (0, 1)$ ,  $q \in [1, \infty)$ , and  $s \in \mathbb{R}$ . Then the  $L^2(X^+)$  inner product (3.9) identifies the dual of  $T_s^{p,q}$  with  $T_{-s;\delta_{1,p}}^{\infty,q'}$ .*

*Proof.* First suppose that  $a$  is a  $T_s^{p,q}$ -atom associated with a ball  $B \subset X$ , and that  $g \in T_{-s,\delta_{1,p}}^{\infty,q'}$ . Then we have

$$\begin{aligned} |\langle a, g \rangle| &\leq \iint_{T(B)} |V^{-s}a(y, t)| |V^s g(y, t)| d\mu(y) \frac{dt}{t} \\ &\leq \|a\|_{T_s^{p,q}} \mu(B)^{1/q'} \mu(B)^{\delta_{1,p}} \|g\|_{T_{-s,\delta_{1,p}}^{\infty,q'}} \\ &\leq \mu(B)^{\delta_{p,q} + \delta_{q,1} + \delta_{1,p}} \|g\|_{T_{-s,\delta_{1,p}}^{\infty,q'}} \\ &= \|g\|_{T_{-s,\delta_{1,p}}^{\infty,q'}}. \end{aligned}$$

For general  $f \in T_s^{p,q}$  we write  $f$  as a sum of  $T_s^{p,q}$ -atoms as in (3.13) and get

$$|\langle f, g \rangle| \leq \|g\|_{T_{-s,\delta_{1,p}}^{\infty,q'}} \|\lambda\|_{\ell^1} \leq \|g\|_{T_{-s,\delta_{1,p}}^{\infty,q'}} \|\lambda\|_{\ell^p}$$

using that  $p < 1$ . Taking the infimum over all atomic decompositions completes the argument.

Conversely, suppose that  $\varphi \in (T_s^{p,q})'$ . Exactly as in the classical duality proof (see [3, Proof of Proposition 3.10]), using the doubling assumption, there exists a function  $g \in L_{\text{loc}}^{q'}(X^+)$  such that

$$\varphi(f) = \langle f, g \rangle$$

for all  $f \in T_s^{p,q;c}$ . To show that  $g$  is in  $T_{-s,\delta_{1,p}}^{\infty,q'}$ , we estimate  $\|V^s g\|_{L^{q'}(T(B))}$  for each ball  $B \subset X$  by duality:

$$\begin{aligned} \|V^s g\|_{L^{q'}(T(B))} &= \sup_{f \in L^q(T(B))} |\langle f, V^s g \rangle| \|f\|_{L^q(T(B))}^{-1} \\ &= \sup_{f \in L_c^q(T(B))} |\langle V^s f, g \rangle| \|f\|_{L^q(T(B))}^{-1}. \end{aligned}$$

Hölder's inequality implies that

$$\|V^s f\|_{T_s^{p,q}} \leq \mu(B)^{\delta_{q,p}} \|f\|_{L^q(T(B))}$$

when  $f$  is essentially supported in  $T(B)$ , so we have

$$\|V^s g\|_{L^{q'}(T(B))} \leq \mu(B)^{\delta_{q,p}} \|\varphi\|_{(T_s^{p,q})'},$$

and therefore

$$\begin{aligned} \|g\|_{T_{-s,\delta_{1,p}}^{\infty,q'}} &= \sup_{B \subset X} \mu(B)^{\delta_{p,1} - (1/q')} \|V^s g\|_{L^{q'}(T(B))} \\ &\leq \|\varphi\|_{(T_s^{p,q})'} \sup_{B \subset X} \mu(B)^{\delta_{p,1} + \delta_{1,q} + \delta_{q,p}} \\ &= \|\varphi\|_{(T_s^{p,q})'}, \end{aligned}$$

which completes the proof.  $\square$

*Remark 3.2.12.* Note that  $q = 1$  is included here, and excluded in the other duality results of this article. Generally the spaces  $T^{p,q}$  with  $p \leq q$  are easier to handle than those with  $p > q$ .

We end this section by detailing a technique, usually referred to as ‘convex reduction’, which is very useful in relating tent spaces to each other. Suppose  $f \in L^0(X^+)$  and  $M > 0$ . We define a function  $f^M \in L^0(X^+)$  by

$$(f^M)(x, t) := |f(x, t)|^M$$

for all  $(x, t) \in X^+$ . For all  $q \in (0, \infty]$  and  $s \in \mathbb{R}$  we then have

$$\mathcal{A}^q(V^{-s}f^M) = \mathcal{A}^{Mq}(V^{-s/M}f)^M,$$

and for  $\alpha \in \mathbb{R}$  we also have

$$\mathcal{C}_\alpha^q(V^{-s}f^M) = \mathcal{C}_{\alpha/M}^{Mq}(V^{-s/M}f)^M.$$

Therefore, for  $p \in (0, \infty)$  we have

$$\begin{aligned} \|f^M\|_{T_s^{p,q}} &= \|\mathcal{A}^{Mq}(V^{-s/M}f)^M\|_{L^p(X)} \\ &= \|\mathcal{A}^{Mq}(V^{-s/M}f)\|_{L^{Mp}(X)}^M \\ &= \|f\|_{T_{s/M}^{Mp,Mq}}^M, \end{aligned}$$

and likewise for  $p = \infty$  and  $q < \infty$  we have

$$\|f^M\|_{T_{s,\alpha}^{\infty,q}} = \|f\|_{T_{s/M,\alpha/M}^{\infty,Mq}}^M$$

The case  $p = q = \infty$  behaves in the same way:

$$\|f^M\|_{T_s^{\infty,\infty}} = \|(V^{-s/M}f)^M\|_{L^\infty(X^+)} = \|f\|_{T_{s/M}^{\infty,\infty}}^M.$$

These equalities often allow us to deduce properties of  $T_s^{p,q}$  from properties of  $T_{s/M}^{Mp,Mq}$ , and vice versa. We will use them frequently.

### 3.3 Interpolation and embeddings

As always, we assume that  $(X, d, \mu)$  is a nondegenerate metric measure space. We will freely use notation and terminology regarding interpolation theory; the uninitiated reader may refer to Bergh and Löfström [22].

#### 3.3.1 Complex interpolation

In this section we will make the following identification of the complex interpolants of weighted tent spaces in the Banach range of exponents.

**Theorem 3.3.1.** *Suppose that  $X$  is doubling,  $p_0, p_1 \in [1, \infty]$  (not both  $\infty$ ),  $q_0, q_1 \in (1, \infty)$ ,  $s_0, s_1 \in \mathbb{R}$ , and  $\theta \in (0, 1)$ . Then we have the identification*

$$[T_{s_0}^{p_0,q_0}, T_{s_1}^{p_1,q_1}]_\theta = T_{s_\theta}^{p_\theta,q_\theta}$$

where  $p_\theta^{-1} = (1 - \theta)p_0^{-1} + \theta p_1^{-1}$ ,  $q_\theta^{-1} = (1 - \theta)q_0^{-1} + \theta q_1^{-1}$ , and  $s_\theta = (1 - \theta)s_0 + \theta s_1$ .

*Remark 3.3.2.* In the case where  $X = \mathbb{R}^n$  with the Euclidean distance and Lebesgue measure, this result (with  $p_0, p_1 < 1$  permitted) is due to Hofmann, Mayboroda, and McIntosh [50, Lemma 8.23]. A more general result, still with  $X = \mathbb{R}^n$ , is proven by Huang [51, Theorem 4.3] with  $q_0, q_1 = \infty$  also permitted, and with Whitney averages incorporated. Both of these results are proven by means of factorisation theorems for weighted tent spaces (with Whitney averages in the second case), and by invoking an extension of Calderón's product formula to quasi-Banach spaces due to Kalton and Mitrea [58, Theorem 3.4]. We have chosen to stay in the Banach range with  $1 < q_0, q_1 < \infty$  for now, as establishing a general factorisation result would take us too far afield.

Note that if  $p_0 = \infty$  (say) then we are implicitly considering  $T_{s_0; \alpha}^{\infty, q_0}$  with  $\alpha = 0$ ; interpolation of spaces with  $\alpha \neq 0$  is not covered by this theorem. This is because the method of proof uses duality, and to realise  $T_{s_0; \alpha}^{\infty, q_0}$  with  $\alpha \neq 0$  as a dual space we would need to deal with complex interpolation of quasi-Banach spaces, which adds difficulties that we have chosen to avoid.

Before moving on to the proof of Theorem 3.3.1, we must fix some notation. For  $q \in (1, \infty)$  and  $s \in \mathbb{R}$ , write

$$L_s^q(X^+) := L^q(X^+, V^{-qs-1}) := L^q\left(X^+, V^{-qs}(y, t) \frac{d\mu(y)}{V(y, t)} \frac{dt}{t}\right) \quad (3.14)$$

(this notation is consistent with viewing the function  $V^{-qs-1}$  as a weight on the product measure  $d\mu dt/t$ ).

An important observation, originating from Harboure, Torrea, and Viviani [44], is that for all  $p \in [1, \infty)$ ,  $q \in (1, \infty)$  and  $s \in \mathbb{R}$ , one can write

$$\|f\|_{T_s^{p, q}} = \left\| Hf \mid L^p(X : L_s^q(X^+)) \right\|$$

for  $f \in L^0(X^+)$ , where

$$Hf(x) = \mathbf{1}_{\Gamma(x)} f.$$

Hence  $H$  is an isometry from  $T_s^{p, q}$  to  $L^p(X : L_s^q(X^+))$ . Because of the restriction on  $q$ , the theory of Lebesgue spaces (more precisely, Bochner spaces) with values in reflexive Banach spaces is then available to us.

This proof follows previous arguments of the author [3], which are based on the ideas of Harboure, Torrea, and Viviani [44] and of Bernal [23], with only small modifications to incorporate additional parameters. We include it to show where these modifications occur: in the use of duality, and in the convex reduction.

*Proof of Theorem 3.3.1.* First we will prove the result for  $p_0, p_1 \in (1, \infty)$ . Since  $H$  is an isometry from  $T_{s_j}^{p_j, q_j}$  to  $L^{p_j}(X : L_{s_j}^{q_j}(X^+))$  for  $j = 0, 1$ , the interpolation property implies that  $H$  is bounded (with norm  $\leq 1$  due to exactness of the complex interpolation functor)

$$[T_{s_0}^{p_0, q_0}, T_{s_1}^{p_1, q_1}]_\theta \rightarrow L^{p_\theta} \left( X : [L_{s_0}^{q_0}(X^+), L_{s_1}^{q_1}(X^+)]_\theta \right).$$

Here we have used the standard identification of complex interpolants of Banach-valued Lebesgue spaces [22, Theorem 5.1.2]. The standard identification of complex interpolants of weighted Lebesgue spaces [22, Theorem 5.5.3] gives

$$[L_{s_0}^{q_0}(X^+), L_{s_1}^{q_1}(X^+)]_\theta = L_{s_\theta}^{q_\theta}(X^+),$$

and we conclude that

$$\begin{aligned} \|f\|_{T_{s_\theta}^{p_\theta, q_\theta}} &= \left\| Hf \mid L^{p_\theta}(X : L_{s_\theta}^{q_\theta}(X^+)) \right\| \\ &\leq \left\| f \mid [T_{s_0}^{p_0, q_0}, T_{s_1}^{p_1, q_1}]_\theta \right\| \end{aligned}$$

for all  $f \in [T_{s_0}^{p_0, q_0}, T_{s_1}^{p_1, q_1}]_\theta$ . Therefore

$$[T_{s_0}^{p_0, q_0}, T_{s_1}^{p_1, q_1}]_\theta \subset T_{s_\theta}^{p_\theta, q_\theta}. \quad (3.15)$$

To obtain the reverse inclusion, we use the duality theorem for complex interpolation [22, Theorem 4.5.1 and Corollary 4.5.2]. Since  $X$  is doubling and by our restrictions on  $p$  and  $q$ , at least one of the spaces  $T_{s_0}^{p_0, q_0}$  and  $T_{s_1}^{p_1, q_1}$  is reflexive (by Proposition 3.2.9) and their intersection is dense in both spaces (as it contains the dense subspace  $L_c^{\max(q_0, q_1)}(X^+)$  by Proposition 3.2.4). Therefore the assumptions of the duality theorem for complex interpolation are satisfied, and we have

$$\begin{aligned} T_{s_\theta}^{p_\theta, q_\theta} &= (T_{-s_\theta}^{p'_\theta, q'_\theta})' \\ &\subset [T_{-s_0}^{p'_0, q'_0}, T_{-s_1}^{p'_1, q'_1}]'_\theta \\ &= [T_{s_0}^{p_0, q_0}, T_{s_1}^{p_1, q_1}]_\theta \end{aligned}$$

where the first two lines follow from Proposition (3.2.9) and (3.15), and the third line uses the duality theorem for complex interpolation combined with Proposition 3.2.9.

We can extend this result to  $p_0, p_1 \in [1, \infty]$  using the technique of [3, Proposition 3.18]. The argument is essentially identical, so we will not include the details here.  $\square$

### 3.3.2 Real interpolation: the reflexive range

In order to discuss real interpolation of weighted tent spaces, we need to introduce a new scale of function spaces, which we denote by  $Z_s^{p,q} = Z_s^{p,q}(X)$ .<sup>3</sup>

**Definition 3.3.3.** For  $c_0 \in (0, \infty)$ ,  $c_1 \in (1, \infty)$ , and  $(x, t) \in X^+$ , we define the *Whitney region*

$$\Omega_{c_0, c_1}(x, t) := B(x, c_0 t) \times (c_1^{-1} t, c_1 t) \subset X^+,$$

and for  $q \in (0, \infty)$ ,  $f \in L^0(X^+)$ , and  $(x, t) \in X^+$  we define the  *$L^q$ -Whitney average*

$$(\mathcal{W}_{c_0, c_1}^q f)(x, t) := \left( \iint_{\Omega_{c_0, c_1}(x, t)} |f(\xi, \tau)|^q d\mu(\xi) d\tau \right)^{1/q}.$$

For  $p, q \in (0, \infty)$ ,  $s \in \mathbb{R}$ ,  $c_0 \in (0, \infty)$ ,  $c_1 \in (1, \infty)$ , and  $f \in L^0(X^+)$ , we then define the quasi-norm

$$\|f\|_{Z_s^{p,q}(X; c_0, c_1)} := \left\| \mathcal{W}_{c_0, c_1}^q (V^{-s} f) \right\|_{L^p(X^+)}.$$

and the  *$Z$ -space*

$$Z_s^{p,q}(X; c_0, c_1) := \{f \in L^0(X^+) : \|f\|_{Z_s^{p,q}(X; c_0, c_1)} < \infty\}.$$

In this section we will prove the following theorem, which identifies real interpolants of weighted tent spaces *in the reflexive range*. We will extend this to the full range of exponents in the Euclidean case in the next section.

**Theorem 3.3.4.** *Suppose that  $X$  is AD-regular and unbounded,  $p_0, p_1, q \in (1, \infty)$ ,  $s_0 \neq s_1 \in \mathbb{R}$ , and  $\theta \in (0, 1)$ . Then for any  $c_0 \in (0, \infty)$  and  $c_1 \in (1, \infty)$  we have the identification*

$$(T_{s_0}^{p_0, q}, T_{s_1}^{p_1, q})_{\theta, p_\theta} = Z_{s_\theta}^{p_\theta, q}(X; c_0, c_1) \quad (3.16)$$

*with equivalent norms, where  $p_\theta^{-1} = (1 - \theta)p_0^{-1} + \theta p_1^{-1}$  and  $s_\theta = (1 - \theta)s_0 + \theta s_1$ .*

As a corollary, in the case when  $X$  is AD-regular and unbounded, and when  $p, q > 1$ , the spaces  $Z_s^{p,q}(X; c_0, c_1)$  are independent of the parameters  $(c_0, c_1)$  with equivalent norms, and we can denote them all simply by  $Z_s^{p,q}$ .<sup>4</sup> We remark that most of the proof does not require AD-regularity, but in its absence we obtain identifications of the real interpolants which are less convenient.

<sup>3</sup>We use this notation because almost every other reasonable letter seems to be taken.

<sup>4</sup>One can prove independence of the parameters  $(c_0, c_1)$  directly when  $X$  is doubling, but proving this here would take us even further off course.

The proof relies on the following identification of real interpolants of weighted  $L^q$  spaces, with fixed  $q$  and distinct weights, due to Gilbert [39, Theorem 3.7]. The cases  $p \leq 1$  and  $q < 1$  are not considered there, but the proof still works without any modifications in these cases. Note that the original statement of this theorem contains a sign error in the expression corresponding to (3.17).

**Theorem 3.3.5** (Gilbert). *Suppose  $(M, \mu)$  is a  $\sigma$ -finite measure space and let  $w$  be a weight on  $(M, \mu)$ . Let  $p, q \in (0, \infty)$  and  $\theta \in (0, 1)$ . For all  $r \in (1, \infty)$ , and for  $f \in L^0(M)$ , the expressions*

$$\left\| \left( r^{-k\theta} \left\| \mathbf{1}_{x:w(x) \in (r^{-k}, r^{-k+1}]} f \right\|_{L^q(M)} \right)_{k \in \mathbb{Z}} \right\|_{\ell^p(\mathbb{Z})} \quad (3.17)$$

$$\left\| s^{1-\theta} \left\| \mathbf{1}_{x:w(x) \leq 1/s} f \right\|_{L^q(M, w^q)} \right\|_{L^p(\mathbb{R}_+, ds/s)} \quad (3.18)$$

and

$$\left\| s^{-\theta} \left\| \mathbf{1}_{x:w(x) > 1/s} f \right\|_{L^q(M)} \right\|_{L^p(\mathbb{R}_+, ds/s)} \quad (3.19)$$

define equivalent norms on the real interpolation space

$$(L^q(M), L^q(M, w^q))_{\theta, p}.$$

The first step in the proof of Theorem 3.3.4 is a preliminary identification of the real interpolation norm.

**Proposition 3.3.6.** *Let all numerical parameters be as in the statement of Theorem 3.3.4. Then for all  $f \in L^0(X^+)$  we have the equivalence*

$$\left\| f \mid (T_{s_0}^{p_0, q}, T_{s_1}^{p_1, q})_{\theta, p_\theta} \right\| \simeq \left\| x \mapsto \left\| \mathbf{1}_{\Gamma(x)} f \mid (L_{s_0}^q(X^+), L_{s_1}^q(X^+))_{\theta, p_\theta} \right\| \right\|_{L^{p_\theta}(X)}. \quad (3.20)$$

*Proof.* We use the notation of the previous section. We have already noted that the map  $H: T_s^{p, q} \rightarrow L^p(X : L_s^q(X^+))$  with  $Hf(x) = \mathbf{1}_{\Gamma(x)} f$  is an isometry. Furthermore, as shown in [3] (see the discussion preceding Proposition 3.12 there),  $H(T_s^{p, q})$  is complemented in  $L^p(X : L_s^q(X^+))$ , and there is a common projection onto these spaces. Therefore we have (by [89, Theorem 1.17.1.1] for example)

$$\left\| f \mid (T_{s_0}^{p_0, q}, T_{s_1}^{p_1, q})_{\theta, p_\theta} \right\| \simeq \left\| Hf \mid (L^{p_0}(X : L_{s_0}^q(X^+)), L^{p_1}(X : L_{s_1}^q(X^+)))_{\theta, p_\theta} \right\|.$$

The Lions–Peetre result on real interpolation of Banach-valued Lebesgue spaces (see for example [76, Remark 7]) then implies that

$$\left\| f \mid (T_{s_0}^{p_0, q}, T_{s_1}^{p_1, q})_{\theta, p_\theta} \right\| \simeq \left\| Hf \mid (L^{p_\theta}(X : (L_{s_0}^q(X^+), L_{s_1}^q(X^+))_{\theta, p_\theta})) \right\|.$$

Since  $Hf(x) = \mathbf{1}_{\Gamma(x)} f$ , this proves (3.20).  $\square$

Having proven Proposition 3.3.6, we can use Theorem 3.3.5 to provide some useful characterisations of the real interpolation norm. For  $f \in L^0(X^+)$  and  $a, b \in [0, \infty]$ , we define the truncation

$$f_{a,b} := \mathbf{1}_{X \times (a,b)} f.$$

Note that in this theorem we allow for  $p_0, p_1 \leq 1$ ; we will use this range of exponents in the next section.

**Theorem 3.3.7.** *Suppose  $p_0, p_1, q \in (0, \infty)$ ,  $s_0 \neq s_1 \in \mathbb{R}$ , and  $\theta \in (0, 1)$ , and suppose that  $X$  is AD-regular of dimension  $n$  and unbounded. Let  $r \in (1, \infty)$ . Then for  $f \in L^0(X^+)$  we have norm equivalences*

$$\begin{aligned} & \left\| x \mapsto \left\| \mathbf{1}_{\Gamma(x)} f \mid (L_{s_0}^q(X^+), L_{s_1}^q(X^+))_{\theta, p_\theta} \right\| \right\|_{L^{p_\theta}(X)} \\ & \simeq \left\| \tau^{n(s_1-s_0)(1-\theta)} \|f_{\tau, \infty}\|_{T_{s_1}^{p_\theta, q}} \right\|_{L^{p_\theta}(\mathbb{R}_+, d\tau/\tau)} \end{aligned} \quad (3.21)$$

$$\simeq \left\| \tau^{-n(s_1-s_0)\theta} \|f_{0, \tau}\|_{T_{s_0}^{p_\theta, q}} \right\|_{L^{p_\theta}(\mathbb{R}_+, d\tau/\tau)} \quad (3.22)$$

$$\simeq_r \left\| (r^{-nk\theta(s_1-s_0)}) \|f_{r^{-k}, r^{-k+1}}\|_{T_{s_0}^{p_\theta, q}} \right\|_{\ell^{p_\theta}(\mathbb{Z})}. \quad (3.23)$$

*Proof.* First assume that  $s_1 > s_0$ . Let  $\mu_{s_0}^q$  be the measure on  $X^+$  given by

$$d\mu_{s_0}^q(y, t) := t^{-qs_0 n} \frac{d\mu(y) dt}{V(y, t)t}.$$

Since  $X$  is AD-regular of dimension  $n$  and unbounded, we have that  $\|f\|_{L^q(\mu_{s_0}^q)} \simeq \|f\|_{L_{s_0}^q(X^+)}$ . Also define the weight  $w(y, t) := t^{-(s_1-s_0)n}$ , so that  $w^q \mu_{s_0}^q = \mu_{s_1}^q$ .

We will obtain the norm equivalence (3.23). For  $1 < r < \infty$  and  $k \in \mathbb{Z}$ , we have  $r^{-k} < w(y, t) \leq r^{-k+1}$  if and only if  $t \in [r^{(k-1)/n(s_1-s_0)}, r^{k/n(s_1-s_0)})$  (here we use  $s_1 > s_0$ ). Using the characterisation (3.17) of Theorem 3.3.5, and replacing  $r$  with  $r^{n(s_1-s_0)}$ , for  $f \in L^0(X^+)$  we have

$$\begin{aligned} & \left\| x \mapsto \left\| \mathbf{1}_{\Gamma(x)} f \mid (L_{s_0}^q(X^+), L_{s_1}^q(X^+))_{\theta, p_\theta} \right\| \right\|_{L^{p_\theta}(X)} \\ & \simeq \left( \int_X \left\| \mathbf{1}_{\Gamma(x)} f \right\|_{(L^q(\mu_{s_0}^q), L^q(w^q \mu_{s_0}^q))_{\theta, p_\theta}}^{p_\theta} d\mu(x) \right)^{1/p_\theta} \\ & \simeq \left( \int_X \sum_{k \in \mathbb{Z}} r^{-n(s_1-s_0)k\theta p_\theta} \left\| \mathbf{1}_{\Gamma(x)} f_{r^{k-1}, r^k} \right\|_{L^q(\mu_{s_0}^q)}^{p_\theta} d\mu(x) \right)^{1/p_\theta} \\ & \simeq \left( \sum_{k \in \mathbb{Z}} r^{-n(s_1-s_0)k\theta p_\theta} \int_X \mathcal{A}^q(V^{-s_0} f_{r^{k-1}, r^k})(x)^{p_\theta} d\mu(x) \right)^{1/p_\theta} \\ & = \left\| (r^{-n(s_1-s_0)k\theta}) \|f_{r^{k-1}, r^k}\|_{T_{s_0}^{p_\theta, q}} \right\|_{\ell^{p_\theta}(\mathbb{Z})}. \end{aligned}$$



This proves the norm equivalence (3.23) for all  $f \in L^0(X^+)$  when  $s_1 > s_0$ . If  $s_1 < s_0$ , one simply uses that  $(L_{s_0}^q(X^+), L_{s_1}^q(X^+))_{\theta, p_\theta} = (L_{s_1}^q(X^+), L_{s_0}^q(X^+))_{1-\theta, p_\theta}$  [22, Theorem 3.4.1(a)] to reduce the problem to the case where  $s_0 < s_1$ .

The equivalences (3.21) and (3.22) follow from the characterisations (3.18) and (3.19) of Theorem 3.3.5 in the same way, with integrals replacing sums throughout. We omit the details here.  $\square$

Finally we can prove the main theorem: the identification of the real interpolants of weighted tent spaces as  $Z$ -spaces.

*Proof of Theorem 3.3.4.* Suppose  $f \in L^0(X^+)$ . Using the characterisation (3.23) in Theorem 3.3.7 with  $r = c_1 > 1$ , and using aperture  $c_0/c_1$  for the tent space (making use of the change of aperture theorem [3, Proposition 3.21]), we have

$$\begin{aligned}
& \left\| f \mid (T_{s_0}^{p_0, q}, T_{s_1}^{p_1, q})_{\theta, p_\theta} \right\|^{p_\theta} \\
& \simeq \sum_{k \in \mathbb{Z}} c_1^{-n(s_1-s_0)k\theta p_\theta} \int_X \left( \int_{c_1^{k-1}}^{c_1^k} \int_{B(x, c_0 t/c_1)} |t^{-ns_0} f(y, t)|^q \frac{d\mu(y)}{V(y, t)} \frac{dt}{t} \right)^{p_\theta/q} d\mu(x) \\
& \simeq \int_X \sum_{k \in \mathbb{Z}} c_1^{-n(s_1-s_0)k\theta p_\theta} \cdot \int_{c_1^{k-1}}^{c_1^k} \left( \int_{c_1^{k-1}}^{c_1^k} \int_{B(x, c_0 t/c_1)} |t^{-ns_0} f(y, t)|^q \frac{d\mu(y)}{V(y, t)} \frac{dt}{t} \right)^{p_\theta/q} \frac{dr}{r} d\mu(x) \\
& \lesssim \int_X \sum_{k \in \mathbb{Z}} c_1^{-n(s_1-s_0)k\theta p_\theta} \int_{c_1^{k-1}}^{c_1^k} \left( \iint_{\Omega_{c_0, c_1}(x, r)} |r^{-ns_0} f(y, t)|^q d\mu(y) dt \right)^{p_\theta/q} \frac{dr}{r} d\mu(x) \\
& \simeq \int_X \int_0^\infty r^{-n(s_1-s_0)\theta p_\theta} \left( \iint_{\Omega_{c_0, c_1}(x, r)} |r^{-ns_0} f|^q \right)^{p_\theta/q} \frac{dr}{r} d\mu(x) \\
& = \iint_{X^+} \left( \iint_{\Omega_{c_0, c_1}(x, r)} |r^{-ns_0} f|^q \right)^{p_\theta/q} d\mu(x) \frac{dr}{r} \\
& \simeq \|f\|_{Z_{s_\theta}^{p_\theta, q}(X; c_0, c_1)}^{p_\theta},
\end{aligned}$$

using that  $B(x, c_0 t/c_1) \times (c_1^{k-1}, c_1^k) \subset \Omega_{c_0, c_1}(x, r)$  whenever  $r \in (c_1^{k-1}, c_1^k)$ .

To prove the reverse estimate we use the same argument, this time using that for  $r, t \in (2^{k-1}, 2^k)$  we have  $\Omega_{c_0, c_1}(x, t) \subset B(x, 2c_0 t) \times (c_1^{-1}2^{k-1}, c_1 2^k)$ . Using

aperture  $2c_0$  for the tent space, we can then conclude that

$$\begin{aligned}
& \|f\|_{Z_{s_\theta}^{p_\theta, q}(X; c_0, c_1)}^{p_\theta} \\
& \simeq \int_X \sum_{k \in \mathbb{Z}} 2^{-n(s_1 - s_0)k\theta p_\theta} \int_{2^{k-1}}^{2^k} \left( \iint_{\Omega_{c_0, c_1}(x, r)} |r^{-ns_0} f|^q \right)^{p_\theta/q} \frac{dr}{r} d\mu(x) \\
& \lesssim \int_X \sum_{k \in \mathbb{Z}} 2^{-n(s_1 - s_0)k\theta p_\theta} \\
& \quad \cdot \int_{2^{k-1}}^{2^k} \left( \int_{c_1^{-1}2^{k-1}}^{c_1 2^k} \int_{B(x, 2c_0 t)} |r^{-ns_0} f(y, t)|^q \frac{d\mu(y)}{V(y, t)} \frac{dt}{t} \right)^{p_\theta/q} \frac{dr}{r} d\mu(x) \\
& \simeq \int_X \sum_{k \in \mathbb{Z}} 2^{-n(s_1 - s_0)k\theta p_\theta} \\
& \quad \cdot \int_{c_1^{-1}2^{k-1}}^{c_1 2^k} \left( \int_{c_1^{-1}2^{k-1}}^{c_1 2^k} \int_{B(x, 2c_0 t)} |r^{-ns_0} f(y, t)|^q \frac{d\mu(y)}{V(y, t)} \frac{dt}{t} \right)^{p_\theta/q} \frac{dr}{r} d\mu(x) \\
& \simeq \|f\|_{(T_{s_0}^{p_0, q}, T_{s_1}^{p_1, q})_{\theta, p_\theta}}^{p_\theta}.
\end{aligned}$$

This completes the proof of Theorem 3.3.4.  $\square$

*Remark 3.3.8.* Note that this argument shows that

$$\left\| (r^{-nk\theta(s_1 - s_0)} \|f_{r^{-k}, r^{-k+1}}\|_{T_{s_0}^{p_\theta, q}})_{k \in \mathbb{Z}} \right\|_{\ell^{p_\theta}(\mathbb{Z})} \simeq \|f\|_{Z_{s_\theta}^{p_\theta, q}(X; c_0, c_1)}$$

whenever  $X$  is AD-regular of dimension  $n$  and unbounded, for all  $p_0, p_1 \in (0, \infty)$ ,  $c_0 \in (0, \infty)$ , and  $c_1 \in (1, \infty)$ . Therefore, since Theorem 3.3.7 also holds for this range of exponents, to establish the identification (3.16) for  $p_0, p_1 \in (0, \infty)$  it suffices to extend Proposition 3.3.6 to  $p_0, p_1 \in (0, \infty)$ . We will do this in the next section in the Euclidean case.

### 3.3.3 Real interpolation: the non-reflexive range

In this section we prove the following extension of Theorem 3.3.4. In what follows, we always consider  $\mathbb{R}^n$  as a metric measure space with the Euclidean distance and Lebesgue measure.

**Theorem 3.3.9.** *Suppose that  $p_0, p_1 \in (0, \infty)$ ,  $q \in [1, \infty)$ ,  $s_0 \neq s_1 \in \mathbb{R}$ , and  $\theta \in (0, 1)$ . Then for any  $c_0 \in (0, \infty)$  and  $c_1 \in (1, \infty)$  we have the identification*

$$(T_{s_0}^{p_0, q}(\mathbb{R}^n), T_{s_1}^{p_1, q}(\mathbb{R}^n))_{\theta, p_\theta} = Z_{s_\theta}^{p_\theta, q}(\mathbb{R}^n; c_0, c_1) \quad (3.24)$$

with equivalent quasi-norms, where  $p_\theta^{-1} = (1-\theta)p_0^{-1} + \theta p_1^{-1}$  and  $s_\theta = (1-\theta)s_0 + \theta s_1$ .

The main difficulty here is that vector-valued Bochner space techniques are not available to us, as we would need to use quasi-Banach valued  $L^p$  spaces with  $p < 1$ , and such a theory is not well-developed. Furthermore, although the weighted tent spaces  $T_s^{p,q}$  embed isometrically into  $L^p(X : L_s^q(X^+))$  in this range of exponents, their image may not be complemented, and so we cannot easily identify interpolants of their images.<sup>5</sup> We must argue directly.

First we recall the so-called ‘power theorem’ [22, Theorem 3.11.6], which allows us to exploit the convexity relations between weighted tent spaces. If  $A$  is a quasi-Banach space with quasi-norm  $||\cdot||$  and if  $\rho > 0$ , then  $||\cdot||^\rho$  is also a quasi-norm on  $A$ , and we denote the resulting quasi-Banach space by  $A^\rho$ .

**Theorem 3.3.10** (Power theorem). *Let  $(A_0, A_1)$  be a compatible couple of quasi-Banach spaces. Let  $\rho_0, \rho_1 \in (0, \infty)$ ,  $\eta \in (0, 1)$ , and  $r \in (0, \infty]$ , and define  $\rho := (1 - \eta)\rho_0 + \eta\rho_1$ ,  $\theta := \eta\rho_1/\rho$ , and  $\sigma := r\rho$ . Then we have*

$$((A_0)^{\rho_0}, (A_1)^{\rho_1})_{\eta, r} = ((A_0, A_1)_{\theta, \sigma})^\rho$$

with equivalent quasi-norms.

Before proving Theorem 3.3.9 we must establish some technical lemmas. Recall that we previously defined the spaces  $L_s^q(X^+)$  in (3.14).

**Lemma 3.3.11.** *Suppose  $x \in X$ ,  $\alpha \in (0, \infty)$ , and let all other numerical parameters be as in the statement of Theorem 3.3.9. Then for all cylindrically supported  $f \in L^0(X^+)$  we have*

$$\begin{aligned} K(\alpha, \mathbf{1}_{\Gamma(x)}f; L_{s_0}^q(X^+), L_{s_1}^q(X^+)) \\ = \inf_{f=\varphi_0+\varphi_1} \left( \mathcal{A}^q(V^{-s_0}\varphi_0)(x) + \alpha \mathcal{A}^q(V^{-s_1}\varphi_1)(x) \right) \end{aligned} \quad (3.25)$$

and

$$\begin{aligned} K(\alpha, \mathbf{1}_{\Gamma(x)}f; L_{s_0}^q(X^+)^{p_0}, L_{s_1}^q(X^+)^{p_1}) \\ = \inf_{f=\varphi_0+\varphi_1} \left( \mathcal{A}^q(V^{-s_0}\varphi_0)(x)^{p_0} + \alpha \mathcal{A}^q(V^{-s_1}\varphi_1)(x)^{p_1} \right) \end{aligned} \quad (3.26)$$

where the infima are taken over all decompositions  $f = \varphi_0 + \varphi_1$  in  $L^0(X^+)$  with  $\varphi_0, \varphi_1$  cylindrically supported.

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<sup>5</sup>Harboure, Torrea, and Viviani [44] avoid this problem by embedding  $T^1$  into a vector-valued Hardy space  $H^1$ . If we were to extend this argument we would need identifications of quasi-Banach real interpolants of certain vector-valued Hardy spaces  $H^p$  for  $p \leq 1$ , which is very uncertain terrain (see Blasco and Xu [25]).

*Proof.* We will only prove the equality (3.25), as the proof of (3.26) is essentially the same.

Given a decomposition  $f = \varphi_0 + \varphi_1$  in  $L^0(X^+)$ , we have a corresponding decomposition  $\mathbf{1}_{\Gamma(x)}f = \mathbf{1}_{\Gamma(x)}\varphi_0 + \mathbf{1}_{\Gamma(x)}\varphi_1$ , with  $\|\mathbf{1}_{\Gamma(x)}\varphi_0\|_{L_{s_0}^q(X^+)} = \mathcal{A}^q(V^{-s_0}\varphi_0)(x)$  and likewise for  $\varphi_1$ . This shows that

$$K(\alpha, \mathbf{1}_{\Gamma(x)}f; L_{s_0}^q(X^+), L_{s_1}^q(X^+)) \leq \inf_{f=\varphi_0+\varphi_1} \left( \mathcal{A}^q(V^{-s_0}\varphi_0)(x) + \alpha \mathcal{A}^q(V^{-s_1}\varphi_1)(x) \right).$$

For the reverse inequality, suppose that  $\mathbf{1}_{\Gamma(x)}f = \varphi_0 + \varphi_1$  in  $L^0(X^+)$ , and suppose  $f$  is essentially supported in a cylinder  $C$ . Multiplication by the characteristic function  $\mathbf{1}_{\Gamma(x) \cap C}$  does not increase the quasi-norms of  $\varphi_0$  and  $\varphi_1$  in  $L_{s_0}^q(X^+)$  and  $L_{s_1}^q(X^+)$  respectively, so without loss of generality we can assume that  $\varphi_0$  and  $\varphi_1$  are cylindrically supported in  $\Gamma(x)$ . Now let  $f = \psi_0 + \psi_1$  be an arbitrary decomposition in  $L^0(X^+)$ , and define

$$\begin{aligned} \widetilde{\psi}_0 &:= \mathbf{1}_{\Gamma(x)}\varphi_0 + \mathbf{1}_{X^+ \setminus \Gamma(x)}\psi_0, \\ \widetilde{\psi}_1 &:= \mathbf{1}_{\Gamma(x)}\varphi_1 + \mathbf{1}_{X^+ \setminus \Gamma(x)}\psi_1. \end{aligned}$$

Then  $f = \widetilde{\psi}_0 + \widetilde{\psi}_1$  in  $L^0(X^+)$ , and we have

$$\mathcal{A}^q(V^{-s_0}\widetilde{\psi}_0)(x) = \mathcal{A}^q(V^{-s_0}\varphi_0)(x) = \|\mathbf{1}_{\Gamma(x)}\varphi_0\|_{L_{s_0}^q(X^+)}$$

and likewise for  $\widetilde{\psi}_1$ . The conclusion follows from the definition of the  $K$ -functional.  $\square$

**Lemma 3.3.12.** *Suppose  $f \in L_c^q(X^+)$ . Then  $\mathcal{A}^q f$  is continuous.*

*Proof.* Let  $f$  be essentially supported in the cylinder  $C := B(c, r) \times (\kappa_0, \kappa_1)$ . First, for all  $x \in X$  we estimate

$$\begin{aligned} \mathcal{A}^q f(x) &\leq \left( \iint_C |f(y, t)|^q \frac{d\mu(y)}{V(y, t)} \frac{dt}{t} \right)^{1/q} \\ &\leq \left( \inf_{y \in B} V(y, \kappa_0) \right)^{-1/q} \|f\|_{L^q(X^+)} \\ &\lesssim \|f\|_{L^q(X^+)}, \end{aligned}$$

using the estimate (3.40) from the proof of Lemma 3.4.1.

For all  $x \in X$  we thus have

$$\lim_{z \rightarrow x} |\mathcal{A}^q f(x) - \mathcal{A}^q f(z)| \leq \lim_{z \rightarrow x} \left( \iint_{X^+} |\mathbf{1}_{\Gamma(x)} - \mathbf{1}_{\Gamma(z)}| |f(y, t)|^q \frac{d\mu(y)}{V(y, t)} \frac{dt}{t} \right)^{1/q} = 0$$

by dominated convergence, since  $\mathbf{1}_{\Gamma(x)} - \mathbf{1}_{\Gamma(z)} \rightarrow 0$  pointwise as  $z \rightarrow x$ , and since

$$\left( \iint_{X^+} |\mathbf{1}_{\Gamma(x)} - \mathbf{1}_{\Gamma(z)}| |f(y, t)|^q \frac{d\mu(y)}{V(y, t)} \frac{dt}{t} \right)^{1/q} \lesssim \|f\|_{L^q(X^+)}.$$

Therefore  $\mathcal{A}^q f$  is continuous.  $\square$

Having established these lemmas, we can prove the following (half-)extension of Proposition 3.3.6.

**Proposition 3.3.13.** *Let all numerical parameters be as in the statement of Theorem 3.3.9. Then for all  $f \in L_c^q(X^+)$  the function*

$$x \mapsto \left\| \mathbf{1}_{\Gamma(x)} f \mid (L_{s_0}^q(X^+), L_{s_1}^q(X^+))_{\theta, p_\theta} \right\| \quad (3.27)$$

is measurable on  $X$  (using the discrete characterisation of the real interpolation quasi-norm), and we have

$$\left\| f \mid (T_{s_0}^{p_0, q}, T_{s_1}^{p_1, q})_{\theta, p_\theta} \right\| \gtrsim \left\| x \mapsto \left\| \mathbf{1}_{\Gamma(x)} f \mid (L_{s_0}^q(X^+), L_{s_1}^q(X^+))_{\theta, p_\theta} \right\| \right\|_{L^{p_\theta}(X)}. \quad (3.28)$$

We denote the quantity on the right hand side of (3.20) by  $\left\| f \mid I_{s_0, s_1, \theta}^{p_\theta, q} \right\|$ .

*Proof.* First we take care of measurability. Using Lemma 3.3.11, for  $x \in X$  we write

$$\begin{aligned} & \left\| \mathbf{1}_{\Gamma(x)} f \mid (L_{s_0}^q(X^+), L_{s_1}^q(X^+))_{\theta, p_\theta} \right\|^{p_\theta} \\ &= \sum_{k \in \mathbb{Z}} 2^{-kp_\theta \theta} K \left( 2^k, \mathbf{1}_{\Gamma(x)} f; L_{s_0}^q(X^+), L_{s_1}^q(X^+) \right)^{p_\theta} \\ &= \sum_{k \in \mathbb{Z}} 2^{-kp_\theta \theta} \inf_{f = \varphi_0 + \varphi_1} \left( \mathcal{A}^q(V^{-s_0} \varphi_0)(x) + 2^k \mathcal{A}^q(V^{-s_1} \varphi_1)(x) \right)^{p_\theta} \end{aligned}$$

where the infima are taken over all decompositions  $f = \varphi_0 + \varphi_1$  in  $L^0(X^+)$  with  $\varphi_0 \in L_{s_0}^q(X^+)$  and  $\varphi_1 \in L_{s_1}^q(X^+)$  cylindrically supported. By Lemma 3.3.12, we have that  $\mathcal{A}^q(V^{-s_0} \varphi_0)$  and  $\mathcal{A}^q(V^{-s_1} \varphi_1)$  are continuous. Hence for each  $k \in \mathbb{Z}$  and for every such decomposition  $f = \varphi_0 + \varphi_1$  the function  $x \mapsto \mathcal{A}^q(V^{-s_0} \varphi_0)(x) + 2^k \mathcal{A}^q(V^{-s_1} \varphi_1)(x)$  is continuous. The infimum of these functions is then upper semicontinuous, therefore measurable.

Next, before beginning the proof of the estimate (3.28), we apply the power theorem with  $A_0 = T_{s_0}^{p_0, q}$ ,  $A_1 = T_{s_1}^{p_1, q}$ ,  $\rho_0 = p_0$ ,  $\rho_1 = p_1$ , and  $\sigma = p_\theta$ . Then we have  $\rho = p_\theta$ ,  $\eta = \theta p_\theta / p_1$ ,  $r = 1$ , and the relation  $p_\theta = (1 - \eta)p_0 + \eta p_1$  is satisfied. We conclude that

$$((T_{s_0}^{p_0, q}, T_{s_1}^{p_1, q})_{\theta, p_\theta})^{p_\theta} \simeq ((T_{s_0}^{p_0, q})^{p_0}, (T_{s_1}^{p_1, q})^{p_1})_{\theta p_\theta / p_1, 1}.$$

Thus it suffices for us to prove

$$\left\| f \mid ((T_{s_0}^{p_0,q})^{p_0}, (T_{s_1}^{p_1,q})^{p_1})_{\theta p_\theta/p_1,1} \right\| \gtrsim \left\| f \mid I_{s_0,s_1,\theta}^{p_\theta,q} \right\|^{p_\theta} \quad (3.29)$$

for all  $f \in L_c^q(X^+)$ .

We write

$$\begin{aligned} & \left\| f \mid ((T_{s_0}^{p_0,q})^{p_0}, (T_{s_1}^{p_1,q})^{p_1})_{\theta p_\theta/p_1,1} \right\| \\ &= \sum_{k \in \mathbb{Z}} 2^{-k\theta p_\theta/p_1} K \left( 2^k, f; (T_{s_0}^{p_0,q})^{p_0}, (T_{s_1}^{p_1,q})^{p_1} \right) \\ &= \sum_{k \in \mathbb{Z}} 2^{-k\theta p_\theta/p_1} \inf_{f=\varphi_0+\varphi_1} \left( \|\varphi_0\|_{T_{s_0}^{p_0,q}}^{p_0} + 2^k \|\varphi_1\|_{T_{s_1}^{p_1,q}}^{p_1} \right) \\ &= \sum_{k \in \mathbb{Z}} 2^{-k\theta p_\theta/p_1} \inf_{f=\varphi_0+\varphi_1} \int_X \mathcal{A}^q(V^{-s_0}\varphi_0)(x)^{p_0} + 2^k \mathcal{A}^q(V^{-s_1}\varphi_1)(x)^{p_1} d\mu(x) \\ &\geq \sum_{k \in \mathbb{Z}} 2^{-k\theta p_\theta/p_1} \int_X \inf_{f=\varphi_0+\varphi_1} \left( \mathcal{A}^q(V^{-s_0}\varphi_0)(x)^{p_0} + 2^k \mathcal{A}^q(V^{-s_1}\varphi_1)(x)^{p_1} \right) d\mu(x) \\ &= \sum_{k \in \mathbb{Z}} 2^{-k\theta p_\theta/p_1} \int_X K \left( 2^k, \mathbf{1}_{\Gamma(x)} f(x); L_{s_0}^q(X^+)^{p_0}, L_{s_1}^q(X^+)^{p_1} \right) d\mu(x) \quad (3.30) \end{aligned}$$

$$\begin{aligned} &= \int_X \left\| \mathbf{1}_{\Gamma(x)} f \mid (L_{s_0}^q(X^+)^{p_0}, L_{s_1}^q(X^+)^{p_1})_{\theta p_\theta/p_1,1} \right\| d\mu(x) \\ &\simeq \int_X \left\| \mathbf{1}_{\Gamma(x)} f \mid (L_{s_0}^q(X^+), L_{s_1}^q(X^+))_{\theta, p_\theta} \right\|^{p_\theta} d\mu(x) \quad (3.31) \\ &= \left\| f \mid I_{s_0,s_1,\theta}^{p_\theta,q} \right\|^{p_\theta} \end{aligned}$$

where again the infima are taken over cylindrically supported  $\varphi_0$  and  $\varphi_1$ . The equality (3.30) is due to Lemma 3.3.11. The equivalence (3.31) follows from the power theorem. This completes the proof of Proposition 3.3.13.  $\square$

As a corollary, we obtain half of the desired interpolation result.

**Corollary 3.3.14.** *Let all numerical parameters be as in the statement of Theorem 3.3.9, and suppose that  $X$  is AD-regular of dimension  $n$  and unbounded.*

$$(T_{s_0}^{p_0,q}, T_{s_1}^{p_1,q})_{\theta, p_\theta} \hookrightarrow Z_{s_\theta}^{p_\theta,q}(X; c_0, c_1). \quad (3.32)$$

*Proof.* This follows from Theorem 3.3.7, Remark 3.3.8, and the density of  $L_c^q(X^+)$  in  $(T_{s_0}^{p_0,q}, T_{s_1}^{p_1,q})_{\theta, p_\theta}$  (which follows from the fact that  $L_c^q(X^+)$  is dense in both  $T_{s_0}^{p_0,q}$  and  $T_{s_1}^{p_1,q}$ , which is due to Lemma 3.2.4).  $\square$

We now prove the reverse containment in the Euclidean case. This rests on a dyadic characterisation of the spaces  $Z_s^{p,q}(\mathbb{R}^n; c_0, c_1)$ . A *standard (open) dyadic*

*cube* is a set  $Q \subset \mathbb{R}^n$  of the form

$$Q = \prod_{i=1}^n (2^k x_i, 2^k(x_i + 1)) \quad (3.33)$$

for some  $k \in \mathbb{Z}$  and  $x \in \mathbb{Z}^n$ . For  $Q$  of the form (3.33) we set  $\ell(Q) := 2^k$  (the sidelength of  $Q$ ), and we denote the set of all standard dyadic cubes by  $\mathcal{D}$ . For every  $Q \in \mathcal{D}$  we define the associated *Whitney cube*

$$\overline{Q} := Q \times (\ell(Q), 2\ell(Q)),$$

and we define  $\mathcal{G} := \{\overline{Q} : Q \in \mathcal{D}\}$ . We write  $\mathbb{R}_+^{n+1} := (\mathbb{R}^n)^+ = \mathbb{R}^n \times (0, \infty)$ . Note that  $\mathcal{G}$  is a partition of  $\mathbb{R}_+^{n+1}$  up to a set of measure zero.

The following proposition is proven by a simple covering argument.

**Proposition 3.3.15.** *Let  $p, q \in (0, \infty)$ ,  $s \in \mathbb{R}$ ,  $c_0 > 0$  and  $c_1 > 1$ . Then for all  $f \in L^0(\mathbb{R}_+^{n+1})$ ,*

$$\|f\|_{Z_s^{p,q}(\mathbb{R}^n; c_0, c_1)} \simeq_{c_0, c_1} \left( \sum_{\overline{Q} \in \mathcal{G}} \ell(Q)^{n(1-ps)} [|f|^q]_{\overline{Q}}^{p/q} \right)^{1/p},$$

where

$$[|f|^q]_{\overline{Q}} := \iint_{\overline{Q}} |f(y, t)|^q dy dt.$$

As a consequence, we gain a convenient embedding.

**Corollary 3.3.16.** *Suppose  $q \in (0, \infty)$ ,  $p \in (0, q]$ , and  $s \in \mathbb{R}$ . Then*

$$Z_s^{p,q}(\mathbb{R}^n) \hookrightarrow T_s^{p,q}(\mathbb{R}^n).$$

*Proof.* We have

$$\begin{aligned}
\|f\|_{T_s^{p,q}(\mathbb{R}^n)} &\simeq \left( \int_{\mathbb{R}^n} \left( \iint_{\Gamma(x)} |t^{-ns} f(y,t)|^q \frac{dy dt}{t^{n+1}} \right)^{p/q} dx \right)^{1/p} \\
&\leq \left( \int_{\mathbb{R}^n} \left( \sum_{\overline{Q} \in \mathcal{G}} \mathbf{1}_{\overline{Q} \cap \Gamma(x) \neq \emptyset}(\overline{Q}) \iint_{\overline{Q}} |t^{-ns} f(y,t)|^q \frac{dy dt}{t^{n+1}} \right)^{p/q} dx \right)^{1/p} \\
&\simeq \left( \int_{\mathbb{R}^n} \left( \sum_{\overline{Q} \in \mathcal{G}} \mathbf{1}_{\overline{Q} \cap \Gamma(x) \neq \emptyset}(\overline{Q}) \ell(Q)^{-nsq} [|f|^q]_{\overline{Q}} \right)^{p/q} dx \right)^{1/p} \\
&\leq \left( \int_{\mathbb{R}^n} \sum_{\overline{Q} \in \mathcal{G}} \mathbf{1}_{\overline{Q} \cap \Gamma(x) \neq \emptyset}(\overline{Q}) \ell(Q)^{-nps} [|f|^q]_{\overline{Q}}^{p/q} dx \right)^{1/p} \tag{3.34}
\end{aligned}$$

$$\begin{aligned}
&= \left( \sum_{\overline{Q} \in \mathcal{G}} \ell(Q)^{-nps} [|f|^q]_{\overline{Q}}^{p/q} |\{x \in \mathbb{R}^n : \Gamma(x) \cap \overline{Q} \neq \emptyset\}| \right)^{1/p} \\
&\lesssim \left( \sum_{\overline{Q} \in \mathcal{G}} \ell(Q)^{n(1-ps)} [|f|^q]_{\overline{Q}}^{p/q} \right)^{1/p} \tag{3.35} \\
&\simeq \|f\|_{Z_s^{p,q}(X; c_0, c_1)},
\end{aligned}$$

where (3.34) follows from  $p/q \leq 1$ , (3.35) follows from

$$|\{x \in \mathbb{R}^n : \Gamma(x) \cap \overline{Q} \neq \emptyset\}| = |B(Q, 2\ell(Q))| \lesssim |Q| \simeq \ell(Q)^n,$$

and the last line follows from Proposition 3.3.15. This proves the claimed embedding.  $\square$

It has already been shown by Barton and Mayboroda that the  $Z$ -spaces form a real interpolation scale [21, Theorem 4.13], in the following sense. We will stop referring to the parameters  $c_0$  and  $c_1$ , as Proposition 3.3.15 implies that the associated quasi-norms are equivalent.

**Proposition 3.3.17.** *Suppose that all numerical parameters are as in the statement of Theorem 3.3.9. Then we have the identification*

$$(Z_{s_0}^{p_0,q}(\mathbb{R}^n), Z_{s_1}^{p_1,q}(\mathbb{R}^n))_{\theta, p_\theta} = Z_{s_\theta}^{p_\theta,q}(\mathbb{R}^n).$$

Now we know enough to complete the proof of Theorem 3.3.9.

*Proof of Theorem 3.3.9.* First suppose that  $p_0, p_1 \in (0, 2]$ . By Corollary 3.3.16 we have

$$Z_{s_j}^{p_j,q}(\mathbb{R}^n) \hookrightarrow T_{s_j}^{p_j,q}(\mathbb{R}^n),$$



for  $j = 0, 1$ , and so

$$(Z_{s_0}^{p_0,q}(\mathbb{R}^n), Z_{s_1}^{p_1,q}(\mathbb{R}^n))_{\theta,p_\theta} \hookrightarrow (T_{s_0}^{p_0,q}(\mathbb{R}^n), T_{s_1}^{p_1,q}(\mathbb{R}^n))_{\theta,p_\theta}.$$

Therefore by Proposition 3.3.17 we have

$$Z_{s_\theta}^{p_\theta,q}(\mathbb{R}^n) \hookrightarrow (T_{s_0}^{p_0,q}(\mathbb{R}^n), T_{s_1}^{p_1,q}(\mathbb{R}^n))_{\theta,p_\theta},$$

and Corollary 3.3.14 then implies that we in fact have equality,

$$Z_{s_\theta}^{p_\theta,q}(\mathbb{R}^n) = (T_{s_0}^{p_0,q}(\mathbb{R}^n), T_{s_1}^{p_1,q}(\mathbb{R}^n))_{\theta,p_\theta}.$$

This equality also holds for  $p_0, p_1 \in (1, \infty)$  by Theorem 3.3.4. By reiteration, this equality holds for all  $p_0, p_1 \in (0, \infty)$ . The proof of Theorem 3.3.9 is now complete.  $\square$

*Remark 3.3.18.* This can be extended to general unbounded AD-regular spaces by establishing a dyadic characterisation along the lines of Proposition 3.3.15 (replacing Euclidean dyadic cubes with a more general system of ‘dyadic cubes’), and then proving analogues of Corollary 3.3.16 and Proposition 3.3.17 using the dyadic characterisation. The Euclidean applications are enough for our planned applications, and the Euclidean argument already contains the key ideas, so we leave further details to any curious readers.

### 3.3.4 Hardy–Littlewood–Sobolev embeddings

In this section we prove the following embedding theorem.

**Theorem 3.3.19** (Weighted tent space embeddings). *Suppose  $X$  is doubling. Let  $0 < p_0 < p_1 \leq \infty$ ,  $q \in (0, \infty]$  and  $s_0 > s_1 \in \mathbb{R}$ . Then we have the continuous embedding*

$$T_{s_0}^{p_0,q} \hookrightarrow T_{s_1}^{p_1,q}$$

*whenever  $s_1 - s_0 = \delta_{p_0,p_1}$ . Furthermore, when  $p_0 \in (0, \infty]$ ,  $q \in (1, \infty)$ , and  $\alpha > 0$ , we have the embedding*

$$T_{s_0}^{p_0,q} \hookrightarrow T_{s_1;\alpha}^{\infty,q}$$

*whenever  $(s_1 + \alpha) - s_0 = \delta_{p_0,\infty}$ .*

These embeddings can be thought of as being of Hardy–Littlewood–Sobolev-type, in analogy with the classical Hardy–Littlewood–Sobolev embeddings of homogeneous Triebel–Lizorkin spaces (see for example [55, Theorem 2.1]).

The proof of Theorem 3.3.19 relies on the following atomic estimate. Note that no geometric assumptions are needed here.

**Lemma 3.3.20.** *Let  $1 \leq p \leq q \leq \infty$  and  $s_0 > s_1 \in \mathbb{R}$  with  $s_1 - s_0 = \delta_{1,p}$ . Suppose that  $a$  is a  $T_{s_0}^{1,q}$ -atom. Then  $a$  is in  $T_{s_1}^{p,q}$ , with  $\|a\|_{T_{s_1}^{p,q}} \leq 1$ .*

*Proof.* Suppose that the atom  $a$  is associated with the ball  $B \subset X$ . When  $p \neq \infty$ , using the fact that  $B(x, t) \subset B$  whenever  $(x, t) \in T(B)$  and that  $-\delta_{1,p} > 0$ , we have

$$\begin{aligned} \|a\|_{T_{s_1}^{p,q}} &= \left\| \mathcal{A}^q(V^{-s_1}a) \right\|_{L^p(B)} \\ &\leq \left\| V^{-\delta_{1,p}} \right\|_{L^\infty(T(B))} \left\| \mathcal{A}^q(V^{-s_0}a) \right\|_{L^p(B)} \\ &\leq \mu(B)^{\delta_{p,1}} \mu(B)^{\delta_{q,p}} \|a\|_{T_{s_0}^{q,q}} \\ &\leq \mu(B)^{\delta_{p,1} + \delta_{q,p} + \delta_{1,q}} \\ &= 1, \end{aligned}$$

where we used Hölder's inequality with exponent  $q/p \geq 1$  in the third line.

When  $p = q = \infty$  the argument is simpler: we have

$$\begin{aligned} \|a\|_{T_{s_1}^{\infty,\infty}} &= \left\| V^{-s_0 - \delta_{1,\infty}} a \right\|_{L^\infty(T(B))} \\ &\leq \left\| V^{-\delta_{1,\infty}} \right\|_{L^\infty(T(B))} \left\| V^{-s_0} a \right\|_{L^\infty(T(B))} \\ &\leq \mu(B)^{\delta_{\infty,1}} \mu(B)^{\delta_{1,\infty}} \\ &= 1 \end{aligned}$$

using the same arguments as before (without needing Hölder's inequality).  $\square$

Now we will prove the embedding theorem. Here is a quick outline of the proof. First we establish the first statement for  $p_0 = 1$  and  $1 < p_1 \leq q$  by using part (1) of Lemma 3.3.20. A convexity argument extends this to  $0 < p_0 < p_1 \leq q$ , with  $q > 1$ . Duality then gives the case  $1 < q \leq p_0 < p_1 \leq \infty$ , including when  $p_1 = \infty$  and  $\alpha \neq 0$ . A composition argument completes the proof with  $q > 1$ . Finally, we use another convexity argument to allow for  $q \in (0, 1]$  (with  $p_1 < \infty$ ). To handle the second statement, we argue by duality again.

*Proof of Theorem 3.3.19.* The proof is split into six steps, corresponding to those of the outline above.

**Step 1.** First suppose that  $f \in T_{s_0}^{1,q}$  and  $1 \leq p_1 \leq q$ . By the weighted atomic decomposition theorem, we can write  $f = \sum_k \lambda_k a_k$  where each  $a_k$  is a  $T_{s_0}^{1,q}$ -atom, with the sum converging in  $T_{s_0}^{1,q}$ . By Lemma 3.3.20 we have

$$\|f\|_{T_{s_1}^{p_1,q}} \leq \|\lambda_k\|_{\ell^1(\mathbb{N})}.$$

Taking the infimum over all atomic decompositions yields the continuous embedding

$$T_{s_0}^{1,q} \hookrightarrow T_{s_1}^{p_1,q} \quad (1 < p_1 \leq q \leq \infty, \quad s_1 - s_0 = \delta_{1,p_1}). \quad (3.36)$$

**Step 2.** Now suppose  $0 < p_0 < p_1 \leq q$ ,  $s_1 - s_0 = \delta_{p_0,p_1}$ , and  $f \in T_{s_0}^{p_0,q}$ . Using (3.36) and noting that  $q/p_0 > 1$  and

$$p_0 s_1 - p_0 s_0 = p_0 \delta_{p_0,p_1} = \delta_{1,p_1/p_0},$$

we have

$$\begin{aligned} \|f\|_{T_{s_1}^{p_1,q}} &= \left\| f^{p_0} \mid T_{p_0 s_1}^{p_1/p_0, q/p_0} \right\|^{1/p_0} \\ &\lesssim \left\| f^{p_0} \mid T_{p_0 s_0}^{1, q/p_0} \right\|^{1/p_0} \\ &= \|f\|_{T_{s_0}^{p_0,q}}, \end{aligned}$$

which yields the continuous embedding

$$T_{s_0}^{p_0,q} \hookrightarrow T_{s_1}^{p_1,q} \quad (0 < p_0 < p_1 \leq q \leq \infty, \quad q > 1, \quad s_1 - s_0 = \delta_{p_0,p_1}). \quad (3.37)$$

**Step 3.** We now use a duality argument. Suppose  $1 < q \leq p_0 < p_1 \leq \infty$ . Define  $\pi_0 := p'_1$ ,  $\pi_1 := p'_0$ ,  $\rho := q'$ ,  $\sigma_0 := -s_1$ , and  $\sigma_1 := -s_0$ , with  $s_1 - s_0 = \delta_{p_0,p_1}$ . Then

$$\sigma_1 - \sigma_0 = -s_0 + s_1 = \delta_{p_0,p_1} = \delta_{\pi_0,\pi_1},$$

and so (3.37) gives the continuous embedding

$$T_{\sigma}^{\pi_0,\rho} \hookrightarrow T_{\sigma_1}^{\pi_1,\rho}.$$

Taking duals results in the continuous embedding

$$T_{s_0}^{p_0,q} \hookrightarrow T_{s_1}^{p_1,q} \quad (1 < q \leq p_0 < p_1 \leq \infty, \quad s_1 - s_0 = \delta_{p_0,p_1}). \quad (3.38)$$

**Step 4.** Now suppose that  $0 < p_0 \leq q \leq p_1 \leq \infty$  and  $q > 1$ , again with  $s_1 - s_0 = \delta_{p_0,p_1}$ . Then combining (3.37) and (3.38) gives continuous embeddings

$$T_{s_0}^{p_0,q} \hookrightarrow T_{s_0+\delta_{p_0,q}}^{q,q} \hookrightarrow T_{s_0+\delta_{p_0,q}+\delta_{q,p_1}}^{p_1,q} = T_{s_1}^{p_1,q}. \quad (3.39)$$

**Step 5.** Finally, suppose  $q \leq 1$ , and choose  $M > 0$  such that  $q/M > 1$ . Then using a similar argument to that of Step 2, with  $M s_1 - M s_0 = M \delta_{p_0,p_1} = \delta_{p_0/M, p_1/M}$ ,

$$\begin{aligned} \|f\|_{T_{s_1}^{p_1,q}} &= \left\| f^M \mid T_{M s_1}^{p_1/M, q/M} \right\|^{1/M} \\ &\lesssim \left\| f^M \mid T_{M s_0}^{p_0/M, q/M} \right\|^{1/M} \\ &= \|f\|_{T_{s_0}^{p_0,q}}. \end{aligned}$$

All possible positions of  $q$  relative to  $0 < p_0 < p_1 \leq \infty$  have thus been covered, so the proof of the first statement is complete.

**Step 6.** For the second statement, we let  $(s_1 + \alpha) - s_0 = \delta_{p_0, \infty}$ , and first we suppose that  $p_0 \in (1, \infty]$ . Let

$$\begin{aligned}\pi_0 &:= (1 + \alpha)^{-1} \in (0, 1), \\ \pi_1 &:= p'_0 \in (1, \infty], \\ \rho &= q', \quad \sigma_0 = -s_1, \quad \sigma_1 = -s_0.\end{aligned}$$

Then  $\alpha = \delta_{1, \pi_0} = \delta_{p_1, \infty}$  and so we have

$$\sigma_1 - \sigma_0 = \delta_{p_0, \infty} - \alpha = \delta_{1, \pi_1} - \delta_{1, \pi_0} = \delta_{\pi_0, \pi_1},$$

which yields

$$T_{\sigma_0}^{\pi_0, \rho} \hookrightarrow T_{\sigma_1}^{\pi_1, \rho}.$$

Taking duals yields

$$T_{s_0}^{p_0, q} \hookrightarrow T_{s_1, \alpha}^{\infty, q},$$

which completes the proof when  $p_0 \in (1, \infty]$ . One last convex reduction argument, as in Step 2, completes the proof.  $\square$

We remark that this technique also yields the embedding  $T_{s_0, \alpha_0}^{\infty, q} \hookrightarrow T_{s_1, \alpha_1}^{\infty, q}$  when  $(s_1 + \alpha_1) - (s_0 + \alpha_0) = 0$ ,  $s_0 > s_1$ , and  $0 \leq \alpha_0 < \alpha_1$ .

*Remark 3.3.21.* The embeddings of Theorems 3.3.19, at least for  $p, q \in (1, \infty)$ , also hold with  $Z_s^{p, q}$  replacing  $T_s^{p, q}$  on either side (or both sides) of the embedding. This can be proven by writing  $Z_s^{p, q}$  as a real interpolation space between tent spaces  $T_{\tilde{s}}^{\tilde{p}, q}$  with  $\tilde{p}$  near  $p$  and  $\tilde{s}$  near  $s$ , applying the tent space embedding theorems, and then interpolating again. These embeddings can also be proven ‘by hand’, even for  $p, q \leq 1$ . We leave the details to any curious readers.

## 3.4 Deferred proofs

### 3.4.1 $T^{p, \infty} - L^\infty$ estimates for cylindrically supported functions

The following lemma, which extends [3, Lemma 3.3] to the case  $q = \infty$ , is used in the proof that  $T^{p, \infty}$  is complete (see Proposition 3.2.4).

**Lemma 3.4.1.** *Suppose that  $X$  is doubling and let  $K \subset X^+$  be cylindrical. Then for all  $p \in [1, \infty]$ ,*

$$\|\mathbf{1}_K f\|_{T^{p,\infty}} \lesssim_K \|f\|_{L^\infty(K)} \lesssim_K \|f\|_{T^{p,\infty}}.$$

*Proof.* When  $p = \infty$  this reduces to

$$\|\mathbf{1}_K f\|_{L^\infty(X^+)} = \|f\|_{L^\infty(K)} \leq \|f\|_{L^\infty(X^+)},$$

which is immediate. Thus it suffices to prove the result for  $p = 1$ , for the general case will then follow by interpolating between the  $L^1(K) \rightarrow L^1(X)$  and  $L^\infty(K) \rightarrow L^\infty(X)$  boundedness of the sublinear operator  $\mathcal{A}^\infty$ . Write  $K \subset B_K \times (\kappa_0, \kappa_1)$  for some ball  $B_K = B(c_K, r_K) \subset X$  and  $0 < \kappa_0 < \kappa_1 < \infty$ .

To prove that  $\|\mathbf{1}_K f\|_{T^{1,\infty}} \lesssim_K \|f\|_{L^\infty(K)}$ , observe that

$$\begin{aligned} \|\mathbf{1}_K f\|_{T^{1,\infty}} &\leq \|f\|_{L^\infty(K)} \mu\{x \in X : \Gamma(x) \cap K \neq \emptyset\} \\ &\leq \|f\|_{L^\infty(K)} V(c_K, r_K + \kappa_1) \end{aligned}$$

because if  $x \notin B(c_K, r_K + \kappa_1)$  then  $\Gamma(x) \cap (B_K \times (\kappa_0, \kappa_1)) = \emptyset$ . Note also that  $V(c_K, r_K + \kappa_1)$  is finite and depends only on  $K$ .

Now we will prove that  $\|f\|_{L^\infty(K)} \lesssim_K \|f\|_{T^{1,\infty}}$ . First note that the doubling property implies that for all  $R > 0$  and for all balls  $B \subset X$ ,

$$\inf_{x \in B} \mu(B(x, R)) \gtrsim_{X,R,r_B} \mu(B). \quad (3.40)$$

Indeed, if  $x \in B$  and  $R \leq 2r_B$  then

$$\mu(B) \leq \mu(B(x, R(2r_B R^{-1}))) \lesssim_X (2r_B R^{-1})^n \mu(B(x, R)).$$

where  $n \geq 0$  is the doubling dimension of  $X$ . If  $R > 2r(B)$  then since  $2r_B R^{-1} < 1$ , we have  $\mu(B) \leq \mu(B(x, R))$ .

Let  $(x_j)_{j \in \mathbb{N}}$  be a countable dense subset of  $B_K$ . Then we have

$$K = \bigcup_{j \in \mathbb{N}} (\Gamma(x_j) + \kappa_0) \cap K.$$

By definition the set  $\{(y, t) \in K : |f(y, t)| > 2^{-1} \|f\|_{L^\infty(K)}\}$  has positive measure, so there exists  $j \in \mathbb{N}$  such that  $|f(y, t)| > 2^{-1} \|f\|_{L^\infty(K)}$  for  $(y, t)$  in some subset of  $(\Gamma(x_j) + \kappa_0) \cap K$  with positive measure. Since  $(\Gamma(x_j) + \kappa_0) \cap K \subset \Gamma(x) \cap K$  for all  $x \in B(x_j, \kappa_0)$ , we have that  $\mathcal{A}^\infty(f)(x) \geq 2^{-1} \|f\|_{L^\infty(K)}$  for all  $x \in B(x_j, \kappa_0)$ .

Therefore, using (3.40),

$$\begin{aligned} \|\mathcal{A}^\infty f\|_{L^1(X)} &\geq \frac{1}{2} \mu(B(x_j, \kappa_0)) \|f\|_{L^\infty(K)} \\ &\gtrsim_{X,K} \mu(B_K) \|f\|_{L^\infty(K)} \\ &\simeq_K \|f\|_{L^\infty(K)}. \end{aligned}$$

This completes the proof of the lemma.  $\square$

### 3.4.2 $T^{p,\infty}$ atomic decomposition

As stated above, the atomic decomposition theorem for  $T^{p,\infty}$  can be proven by combining the arguments of Coifman–Meyer–Stein (who prove the result in the Euclidean case) and Russ (who proves the atomic decomposition of  $T^{p,2}(X)$  for  $0 < p \leq 1$  when  $X$  is doubling).

First we recall a classical lemma (see for example [81, Lemma 2.2]), which combines a Vitali-type covering lemma with a partition of unity. This is proven by combining the Vitali-type covering of Coifmann–Weiss [34, Théorème 1.3] with the partition of unity of Macías–Segovia [65, Lemma 2.16].

**Lemma 3.4.2.** *Suppose that  $X$  is doubling, and let  $O$  be a proper subset of  $X$  of finite measure. For all  $x \in X$  write  $r(x) := \text{dist}(x, O^c)/10$ . Then there exists  $M > 0$ , a countable indexing set  $I$ , and a collection of points  $\{x_i\}_{i \in I}$  such that*

- $O = \cup_{i \in I} B(x_i, r(x_i))$ ,
- if  $i, j \in I$  are not equal, then  $B(x_i, r(x_i)/4)$  and  $B(x_j, r(x_j)/4)$  are disjoint, and
- for all  $i \in I$ , there exist at most  $M$  indices  $j \in I$  such that  $B(x_j, 5r(x_j))$  meets  $B(x_i, 5r(x_i))$ .

Moreover, there exist a collection of measurable functions  $\{\varphi_i: X \rightarrow [0, 1]\}_{i \in I}$  such that

- $\text{supp } \varphi_i \subset B(x_i, 2r(x_i))$ ,
- $\sum_i \varphi_i = \mathbf{1}_O$  (for each  $x \in X$  the sum  $\sum_i \varphi_i(x)$  is finite due to the third condition above).

Now we can follow a simplified version of the argument of Russ, which is essentially the argument of Coifman–Meyer–Stein with the partition of unity of Lemma 3.4.2 replacing the use of the Whitney decomposition.

*Proof of Theorem 3.2.6, with  $q = \infty$ .* Suppose  $f \in T^{p,\infty}$ , and for each  $k \in \mathbb{Z}$  define the set

$$O^k := \{x \in X : \mathcal{A}^\infty f(x) > 2^k\}.$$

The sets  $O^k$  are open by lower semicontinuity of  $\mathcal{A}^\infty f$  (Lemma 3.2.1), and the function  $f$  is essentially supported in  $\cup_{k \in \mathbb{Z}} T(O_k) \setminus T(O_{k+1})$ . Thus we can write

$$f = \sum_{k \in \mathbb{Z}} \mathbf{1}_{T(O_k) \setminus T(O_{k+1})} f. \quad (3.41)$$

**Case 1:**  $\mu(X) = \infty$ . In this case we must have  $\mu(O^k) < \infty$  for each  $k \in \mathbb{Z}$ , for otherwise we would have  $\|\mathcal{A}^\infty f\|_{L^p(X)} = \infty$  and thus  $f \notin T^{p,\infty}$ . Hence for each  $k \in \mathbb{Z}$  there exist countable collections of points  $\{x_i^k\}_{i \in I^k} \subset O^k$  and measurable functions  $\{\varphi_i^k\}_{i \in I^k}$  as in Lemma 3.4.2. Combining (3.41) with  $\sum_{i \in I^k} \varphi_i^k = \mathbf{1}_{O^k}$  and  $T(O^k) \subset O^k \times \mathbb{R}_+$ , we can write

$$\begin{aligned} f(y, t) &= \sum_{k \in \mathbb{Z}} \sum_{i \in I^k} \varphi_i^k(y) \mathbf{1}_{T(O^k) \setminus T(O_{k+1})}(y, t) f(y, t) \\ &= \sum_{k \in \mathbb{Z}} \sum_{i \in I^k} \tilde{a}_i^k(y, t). \end{aligned}$$

Note that

$$\|\tilde{a}_i^k\|_{L^\infty(X^+)} \leq \operatorname{ess\,sup}_{(y,t) \notin T(O_{k+1})} |f(y, t)| \leq 2^{k+1}, \quad (3.42)$$

the second inequality following from  $T(O_{k+1}) = X^+ \setminus (\cup_{x \notin O_{k+1}} \Gamma(x))$  and the fact that  $|f(y, t)| \leq \mathcal{A}^\infty f(x) \leq 2^{k+1}$  for all  $x \notin O_{k+1}$  and  $(y, t) \in \Gamma(x)$ .

Define

$$a_i^k := 2^{-(k+1)} \mu(B_i^k)^{-1/p} \tilde{a}_i^k,$$

where  $B_i^k := B(x_i^k, 14r(x_i^k))$ . We claim that  $a_i^k$  is a  $T^{p,\infty}$ -atom associated with the ball  $B_i^k$ . The estimate (3.42) immediately implies the size condition

$$\|a_i^k\|_{T^{\infty,\infty}} \leq \mu(B_i^k)^{\delta_{p,\infty}},$$

so we need only show that  $a_i^k$  is essentially supported in  $T(B_i^k)$ . To show this, it is sufficient to show that if  $y \in B(x_i^k, 2r(x_i^k))$  and  $d(y, (O^k)^c) \geq t$ , then  $d(y, (B_i^k)^c) \geq t$ . Suppose  $z \notin B_i^k$  (such a point exists because  $\mu(B_i^k) < \mu(X) = \infty$ ),  $\varepsilon > 0$  and  $u \notin O^k$  such that

$$d(x_i^k, u) < d(x_i^k, (O^k)^c) + \varepsilon = 10r(x_i^k) + \varepsilon.$$

Then we have

$$\begin{aligned}
d(y, z) + \varepsilon &\geq d(z, x_i^k) - d(x_i^k, y) + \varepsilon \\
&\geq 12r(x_i^k) + \varepsilon \\
&= 2r(x_i^k) + 10r(x_i^k) + \varepsilon \\
&> d(y, x_i^k) + d(x_i^k, u) \\
&\geq d(y, u) \\
&\geq t,
\end{aligned}$$

where the last line follows from  $u \notin O^k$  and  $d(y, (O^k)^c) \geq t$ . Since  $z \notin B_i^k$  and  $\varepsilon > 0$  were arbitrary, this shows that  $d(y, (B_i^k)^c) \geq t$  as required, which proves that  $a_i^k$  is a  $T^{p,\infty}$ -atom associated with  $B_i^k$ .

Thus we have

$$f(y, t) = \sum_{k \in \mathbb{Z}} \sum_{i \in I^k} \lambda_i^k a_i^k,$$

where

$$\lambda_i^k = 2^{k+1} \mu(B_i^k)^{1/p}.$$

It only remains to show that

$$\sum_{k \in \mathbb{Z}} \sum_{i \in I^k} |\lambda_i^k|^p \lesssim \|f\|_{T^{p,\infty}}^p.$$

We estimate

$$\begin{aligned}
\sum_{k \in \mathbb{Z}} \sum_{i \in I^k} |\lambda_i^k|^p &= \sum_{k \in \mathbb{Z}} 2^{(k+1)p} \sum_{i \in I^k} \mu(B_i^k) \\
&\lesssim_X \sum_{k \in \mathbb{Z}} 2^{(k+1)p} \sum_{i \in I^k} \mu(B(x_i^k, r(x_i^k)/4)) \tag{3.43}
\end{aligned}$$

$$\leq \sum_{k \in \mathbb{Z}} 2^{(k+1)p} \mu(O^k) \tag{3.44}$$

$$\begin{aligned}
&\lesssim p \sum_{k \in \mathbb{Z}} \int_{2^{k-1}}^{2^k} t^{p-1} \mu(\{x \in X : \mathcal{A}^\infty f(x) > t\}) dt \\
&= \|\mathcal{A}^\infty f\|_{L^p(X)}^p \\
&= \|f\|_{T^{p,\infty}}^p,
\end{aligned}$$

using doubling in (3.43) and pairwise disjointness of the balls  $B(x_i^k, r(x_i^k)/4)$  in (3.44). This completes the proof in the case that  $\mu(X) = \infty$ .

**Case 2:**  $\mu(X) < \infty$ . In this case we may have  $O^k = X$  for some  $k \in \mathbb{Z}$ , so we cannot apply Lemma 3.4.2 as before. One can follow the argument of Russ



[81, page 131], which shows that the partition of unity is not required for such  $k$ . With this modification, the argument of the previous case still works. We omit the details.  $\square$



## Part II

# Abstract Hardy–Sobolev and Besov spaces for elliptic boundary value problems with complex $L^\infty$ coefficients



## Abstract

We establish a theory of Besov–Hardy–Sobolev spaces adapted to operators which are bisectorial on  $L^2$ , with bounded  $H^\infty$  functional calculus on their ranges, and satisfying off-diagonal estimates. We apply these spaces to the study of well-posedness of boundary value problems associated with elliptic systems  $\operatorname{div} A \nabla u = 0$  with complex  $t$ -independent coefficients on the upper half-space, and with boundary data in classical Besov–Hardy–Sobolev spaces.

In the range of exponents for which the Besov–Hardy–Sobolev spaces adapted to the perturbed Dirac operator  $DB$  are equal to those adapted to the unperturbed operator  $D$  (where  $B$  is a bounded multiplier associated with  $A$ ), we show that well-posedness of a boundary value problem is equivalent to an associated projection being an isomorphism. This is done by classifying all solutions to Cauchy–Riemann systems associated with  $DB$ , or equivalently all conormal gradients to solutions of  $\operatorname{div} A \nabla u = 0$ , within certain weighted tent spaces and their real interpolants. Our approach uses minimal assumptions on the coefficients  $A$ , and in particular does not require De Giorgi–Nash–Moser estimates.

As an application, for real coefficient scalar equations, we extend known well-posedness results for the Regularity problem with data in Hardy and Lebesgue spaces to a large range of Besov–Hardy–Sobolev spaces by interpolation and duality.



# Chapter 4

## Introduction

### 4.1 Introduction and context

This main focus of this article is the well-posedness of boundary value problems associated to divergence-form elliptic systems

$$L_A u := \operatorname{div} A \nabla u = 0, \quad (4.1)$$

where the unknown is a  $\mathbb{C}^m$ -valued function  $u$  on the upper half-space  $\mathbb{R}_+^{1+n} := \{(t, x) \in \mathbb{R}^{1+n} : t > 0\}$ . We work in ambient dimension  $1 + n \geq 2$ , with  $m \geq 1$ . The special case  $m = 1$  corresponds to a scalar equation rather than a system.

The gradient operator  $\nabla$  sends  $\mathbb{C}^m$ -valued functions  $f$  to  $\mathbb{C}^{m(1+n)}$ -valued functions ( $\mathbb{C}^m$ -valued vector fields)  $\nabla f$  by considering  $f = (f^j)_{j=1}^m$  as an  $m$ -tuple of  $\mathbb{C}$ -valued functions, and acting as the usual gradient operator componentwise. The divergence operator  $\operatorname{div}$  is defined similarly, sending  $\mathbb{C}^{m(1+n)}$ -valued functions to  $\mathbb{C}^m$ -valued functions. These differential operators are interpreted in the weak (distributional) sense. Vectors  $v \in \mathbb{C}^{m(1+n)}$  are split into *transversal* and *tangential* parts  $v = (v_\perp, v_\parallel)$  according to the splitting

$$\mathbb{C}^{m(1+n)} = \mathbb{C}^m \oplus \mathbb{C}^{mn}, \quad (4.2)$$

and likewise functions  $f$  with codomain  $\mathbb{C}^{m(1+n)}$  can be split into transversal and tangential parts  $f = (f_\perp, f_\parallel)$ , with codomains  $\mathbb{C}^m$  and  $\mathbb{C}^{mn}$  respectively. We write  $\nabla_\parallel$  and  $\operatorname{div}_\parallel$  for the corresponding tangential restrictions of  $\nabla$  and  $\operatorname{div}$ .

Throughout the entire article we assume (unless explicitly stated otherwise) that the coefficient matrix  $A \in L^\infty(\mathbb{R}_+^{n+1} : \mathcal{L}(\mathbb{C}^{m(1+n)}))$  is bounded, measurable, complex, and *t-independent*, meaning that  $A(t, x) = A(x)$  for almost every  $(t, x) \in \mathbb{R}_+^{1+n}$ . Thus we may identify  $A$  as an element of  $L^\infty(\mathbb{R}^n : \mathcal{L}(\mathbb{C}^{m(1+n)}))$ .

Furthermore we assume that  $A$  is *strictly accretive on curl-free vector fields*, in the sense that there exists  $\kappa > 0$  such that

$$\operatorname{Re} \int_{\mathbb{R}^n} (A(x)f(x), f(x)) \, dx \geq \kappa \|f\|_2^2 \quad (4.3)$$

for all  $f \in L^2(\mathbb{R}^n : \mathbb{C}^{m(1+n)})$  such that  $\operatorname{curl}_{\parallel}(f_{\parallel}) = 0$ . The round bracket in the integrand above is the usual Hermitean inner product on  $\mathbb{C}^{m(1+n)}$ . By  $\operatorname{curl}_{\parallel}(f_{\parallel}) = 0$  we mean that

$$\partial_j f_k = \partial_k f_j \quad (1 \leq k, j \leq n, \quad k \neq j),$$

with (weak) partial derivatives acting componentwise on  $\mathbb{C}^m$ -valued functions. The strict accretivity condition (4.3) is weaker than the usual notion of pointwise strict accretivity

$$\operatorname{Re}(A(x)v, v) \geq \kappa |v|^2 \quad (v \in \mathbb{C}^{m(1+n)}, \quad x \in \mathbb{R}^n)$$

unless  $m = 1$ , in which case these two notions are equivalent (see [8, §2]).

We always consider weak solutions to (4.1). That is, we say that a function  $u \in W_{1,\text{loc}}^2(\mathbb{R}^n : \mathbb{C}^m)$  solves (4.1) if for all  $\varphi \in C_0^\infty(\mathbb{R}_+^{1+n} : \mathbb{C}^m)$  we have

$$\iint_{\mathbb{R}_+^{1+n}} (A(x)\nabla u(t, x), \nabla \varphi(t, x)) \, dx \, dt = 0.$$

#### 4.1.1 Formulation of boundary value problems

One can formulate various boundary value problems associated with the equation  $L_A u = 0$ . First, for  $1 < p < \infty$ , we formulate the  $L^p$ -Dirichlet problem for  $L_A$ , denoted by  $(D_H)_{0,A}^p$ :

$$(D_H)_{0,A}^p : \begin{cases} L_A u = 0 & \text{in } \mathbb{R}_+^{1+n}, \\ \lim_{t \rightarrow 0} u(t, \cdot) = f \in L^p(\mathbb{R}^n : \mathbb{C}^m), \\ \|N_* u\|_{L^p} \lesssim \|f\|_{L^p}, \end{cases}$$

This should be read:

for all  $f \in L^p(\mathbb{R}^n : \mathbb{C}^m)$ ,  
there exists  $u \in W_{1,\text{loc}}^2(\mathbb{R}_+^{1+n} : \mathbb{C}^m)$  solving  $L_A u = 0$ ,  
with  $u \rightarrow f$  in  $L^p$  (the *boundary condition*),  
such that  $\|N_* u\|_p \lesssim \|f\|_p$  (the *interior estimate*).

Here  $N_*$  is the non-tangential maximal function

$$N_* u(x) := \sup_{(t,y) \in \Gamma(x)} |u(t, y)|,$$



where  $\Gamma(x)$  is the cone in  $\mathbb{R}_+^{1+n}$  based at  $x$  (defined in Subsection 5.1.2). We say that the problem  $(D_H)_{0,A}^p$  is *well-posed* if for all  $f \in L^p(\mathbb{R}^n : \mathbb{C}^m)$  there exists a unique  $u$  satisfying these conditions.

For all of the boundary value problems that we consider, well-posedness is defined analogously: for all boundary data, there must exist a unique solution (modulo constants, for Regularity and Neumann problems) which satisfies the stated conditions.

Next, for  $n/(n+1) < p < \infty$ , we formulate the  $H^p$ -Regularity problem for  $L_A$ :

$$(R_H)_{0,A}^p : \begin{cases} L_A u = 0 & \text{in } \mathbb{R}_+^{1+n}, \\ \lim_{t \rightarrow 0} \nabla_{\parallel} u(t, \cdot) = \nabla_{\parallel} f \in H^p(\mathbb{R}^n : \mathbb{C}^{mn}), \\ \|\widetilde{N}_*(\nabla u)\|_{L^p} \lesssim \|\nabla_{\parallel} f\|_{H^p}, \end{cases}$$

where  $\widetilde{N}_* u$  is the modified non-tangential maximal function

$$\widetilde{N}_* u(x) := \sup_{(t,y) \in \Gamma(x)} \left( \iint_{\Omega(t,y)} |u(\tau, \xi)|^2 d\tau d\xi \right)^{1/2} \quad (4.4)$$

(the Whitney region  $\Omega(t, y)$  is defined in Subsection 5.1.3), and where  $H^p(\mathbb{R}^n : \mathbb{C}^{mn})$  is the  $(\mathbb{C}^{mn}$ -valued) real Hardy space, which may be identified with  $L^p(\mathbb{R}^n : \mathbb{C}^{mn})$  when  $p > 1$ .

*Remark 4.1.1.* If  $f$  is a distribution with  $\nabla_{\parallel} f \in H^p(\mathbb{R}^n : \mathbb{C}^{mn})$ , then  $f$  may be identified with an element of  $\dot{H}_1^p(\mathbb{R}^n : \mathbb{C}^m)$  (the  $\mathbb{C}^m$ -valued homogeneous Hardy–Sobolev space of order 1, defined in Subsection 5.1.5), and the boundary condition  $\lim_{t \rightarrow 0} \nabla_{\parallel} u(t, \cdot) = \nabla_{\parallel} f \in H^p(\mathbb{R}^n : \mathbb{C}^{mn})$  is equivalent to the condition

$$\lim_{t \rightarrow 0} u(t, \cdot) = f \in \dot{H}_1^p(\mathbb{R}^n : \mathbb{C}^m).$$

Therefore, by considering potentials rather than tangential gradients, we can see the  $H^p$ -Regularity problem as a kind of  $\dot{H}_1^p$ -Dirichlet problem. Conversely, by shifting viewpoint from functions to their tangential gradients, the  $L^p$ -Dirichlet problem  $(D_H)_{0,A}^p$  can be seen as a kind of  $\dot{H}_{-1}^p$ -Regularity problem. It will be technically convenient for us to consider Regularity problems rather than Dirichlet problems.

For  $n/(n+1) < p < \infty$ , we also formulate the  $H^p$ -Neumann problem for  $L_A$ ,

$$(N_H)_{0,A}^p : \begin{cases} L_A u = 0 & \text{in } \mathbb{R}_+^{1+n}, \\ \lim_{t \rightarrow 0} \partial_{\nu_A} u(t, \cdot) = \partial_{\nu_A} f \in H^p(\mathbb{R}^n : \mathbb{C}^m), \\ \|\widetilde{N}_*(\nabla u)\|_{L^p} \lesssim \|\partial_{\nu_A} f\|_{H^p}, \end{cases}$$

where the  $A$ -conormal derivative  $\partial_{\nu_A} u$  of  $u$  is given by

$$\partial_{\nu_A} u(t, \cdot) = -e_0 \cdot A \nabla u(\cdot, t), \quad (4.5)$$

where  $-e_0$  is the normal vector to  $\mathbb{R}^n \subset \mathbb{R}^{1+n}$ , relative to  $\mathbb{R}_+^{1+n}$ .

The boundary value problems  $(D_H)_{0,A}^p$ ,  $(R_H)_{0,A}^p$ , and  $(N_H)_{0,A}^p$  are all problems of *order zero*:<sup>1</sup> in each of these problems, the interior estimates are in terms of boundary data in either the Lebesgue space  $L^p$  or the Hardy space  $H^p$ . One can also formulate Regularity and Neumann problems of *order*  $-1$ .

For  $1 < p < \infty$ , the  $\dot{H}_{-1}^p$ -Regularity problem, which is similar to the  $L^p$ -Dirichlet problem but with a different interior estimate<sup>2</sup> and a decay condition at infinity (see Remark 4.1.1), is

$$(R_H)_{-1,A}^p : \begin{cases} L_A u = 0 & \text{in } \mathbb{R}_+^{1+n}, \\ \lim_{t \rightarrow 0} \nabla_{\parallel} u(t, \cdot) = \nabla_{\parallel} f \in \dot{H}_{-1}^p(\mathbb{R}^n : \mathbb{C}^{mn}), \\ \lim_{t \rightarrow \infty} \nabla_{\parallel} u(t, \cdot) = 0 & \text{in } \mathcal{Z}'(\mathbb{R}^n : \mathbb{C}^{mn}) \\ \|\nabla u\|_{T_{-1}^p} \lesssim \|\nabla_{\parallel} f\|_{\dot{H}_{-1}^p}. \end{cases}$$

Here  $\mathcal{Z}'(\mathbb{R}^n : \mathbb{C}^{mn})$  is the space of  $\mathbb{C}^{mn}$ -valued tempered distributions modulo polynomials; this is the natural space in which all homogeneous Hardy–Sobolev and Besov spaces are embedded. We can enlarge the range of exponents to ‘ $p \geq \infty$ ’; this is done rigorously by using BMO and the homogeneous Hölder spaces  $\dot{\Lambda}_{\alpha}$ . For  $0 < \alpha < 1$  we define

$$(R_H)_{-1,A}^{(\infty,\alpha)} : \begin{cases} L_A u = 0 & \text{in } \mathbb{R}_+^{1+n}, \\ \lim_{t \rightarrow 0} \nabla_{\parallel} u(t, \cdot) = \nabla_{\parallel} f \in \dot{\Lambda}_{\alpha-1}(\mathbb{R}^n : \mathbb{C}^{mn}), \\ \lim_{t \rightarrow \infty} \nabla_{\parallel} u(t, \cdot) = 0 & \text{in } \mathcal{Z}'(\mathbb{R}^n : \mathbb{C}^{mn}) \\ \|\nabla u\|_{T_{-1;\alpha}^{\infty}} \lesssim \|\nabla_{\parallel} f\|_{\dot{\Lambda}_{\alpha-1}}. \end{cases}$$

and furthermore, with  $\alpha = 0$ ,

$$(R_H)_{-1,A}^{(\infty,0)} : \begin{cases} L_A u = 0 & \text{in } \mathbb{R}_+^{1+n}, \\ \lim_{t \rightarrow 0} \nabla_{\parallel} u(t, \cdot) = \nabla_{\parallel} f \in \dot{\text{BMO}}_{-1}(\mathbb{R}^n : \mathbb{C}^{mn}), \\ \lim_{t \rightarrow \infty} \nabla_{\parallel} u(t, \cdot) = 0 & \text{in } \mathcal{Z}'(\mathbb{R}^n : \mathbb{C}^{mn}) \\ \|\nabla u\|_{T_{-1;0}^{\infty}} \lesssim \|\nabla_{\parallel} f\|_{\dot{\text{BMO}}_{-1}}. \end{cases}$$

<sup>1</sup>We use the term ‘order’ here, since there is no confusion with this ‘order’ and the fact that these are boundary value problems for ‘second-order’ elliptic equations. We could use the term ‘regularity’ instead, but this would probably cause more ambiguity with the Regularity problem.

<sup>2</sup>It is known that  $\|\tilde{N}_* u\|_{L^p} \lesssim \|\nabla u\|_{T_{-1}^p}$  in the range of  $p$  that we shall deal with, but the converse is in general not known.

The spaces  $\dot{\text{BMO}}_{-1}$  and  $\dot{\Lambda}_{\alpha-1}$  are best considered as the homogeneous Triebel–Lizorkin space  $\dot{F}_{-1}^{\infty,2}$  and Besov spaces  $\dot{B}_{\alpha-1}^{\infty,\infty}$  respectively, as their negative orders prevent traditional (i.e. non-Littlewood–Paley) characterisations in terms of smoothness. For these problems the limit in the boundary condition is imposed in the weak-star topology.

With the same ranges of  $p$  and  $\alpha$ , we also define order  $-1$  Neumann problems  $(N_H)_{-1,A}^p$ ,  $(N_H)_{-1,A}^{(\infty,\alpha)}$ , and  $(N_H)_{-1,A}^{(\infty,0)}$  in the same way, with tangential gradients  $\nabla_{\parallel}$  replaced by  $A$ -conormal derivatives  $\partial_{\nu_A}$  in the boundary condition (we keep  $\nabla_{\parallel}$  in the decay condition at infinity).

Note that in the ‘order  $-1$ ’ problems above, we impose a tent space estimate on  $\nabla u$  rather than a nontangential maximal function estimate. The weighted tent spaces  $T_{-1}^p$  and  $T_{-1;0}^{\infty}$  are defined in Subsection 5.1.2. We also impose a decay condition on the tangential gradient  $\nabla_{\parallel} u$  at infinity. For  $p$  sufficiently small this is implied by the other conditions; we remark that if  $L_A$  satisfies a De Giorgi–Nash–Moser condition (see (7.51)) then it is implied for all  $p < \infty$ , and also for some range of  $\alpha > 0$ . (see Lemma 7.2.1).

*Remark 4.1.2.* We have not imposed any nontangential convergence of solutions to boundary data in the problems above. This is because the classification theorems of Auscher and Mourgoglou, in particular [15, Corollaries 1.2 and 1.4], automatically yield almost everywhere (a.e.) non-tangential convergence of Whitney averages (of either the solution or its conormal gradient, whichever is relevant) to the boundary data. When the operator  $L_A$  satisfies a De Giorgi–Nash–Moser condition (see (7.51)) this can be improved to a.e. non-tangential convergence without Whitney averages.

Let us summarise the problems we have introduced so far. There are Dirichlet problems of order 0 and 1 (seeing the  $H^p$ -Regularity problem as a  $\dot{H}_1^p$ -Dirichlet problem), Regularity problems of order 0 and  $-1$ , and Neumann problems of order 0 and  $-1$ .

In their recent monograph [21], Barton and Mayboroda consider problems of *intermediate order*. They formulate Dirichlet problems of order  $\theta \in (0, 1)$  and Neumann problems of order  $\theta \in (-1, 0)$  as follows.<sup>3</sup> For  $0 < \theta < 1$  and

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<sup>3</sup>We have a different indexing convention, where we index our problems according to the order of the boundary function space used in the interior estimate. Barton and Mayboroda refer to  $(N_B)_{\theta-1,A}^p$  as  $(N)_{\theta,A}^p$ . Also, Barton and Mayboroda only consider scalar equations, i.e. the case  $m = 1$ .

$$n/(n + \theta) < p \leq \infty,$$

$$(D_B)_{\theta,A}^p : \begin{cases} L_A u = 0 & \text{in } \mathbb{R}_+^{1+n}, \\ \text{Tr } u = f \in \dot{B}_\theta^{p,p}(\mathbb{R}^n : \mathbb{C}^m) \\ \|\nabla u\|_{L(p,\theta,2)} \lesssim \|f\|_{\dot{B}_\theta^{p,p}}, \end{cases}$$

and

$$(N_B)_{\theta-1,A}^p : \begin{cases} L_A u = 0 & \text{in } \mathbb{R}_+^{1+n}, \\ \partial_{\nu_A} u|_{\partial\mathbb{R}_+^{1+n}} = \partial_{\nu_A} f \in \dot{B}_{\theta-1}^{p,p}(\mathbb{R}^n : \mathbb{C}^m) \\ \|\nabla u\|_{L(p,\theta,2)} \lesssim \|\partial_{\nu_A} f\|_{\dot{B}_{\theta-1}^{p,p}}. \end{cases}$$

The Besov spaces  $\dot{B}_\theta^{p,p}$  are defined in Subsection 5.1.5. The spaces  $L(p, \theta, 2)$  are defined by the norms

$$\|F\|_{L(p,\theta,2)} := \left( \iint_{\mathbb{R}_+^{1+n}} \left( \iint_{\Omega(t,x)} |\tau^{1-\theta} F(\tau, \xi)|^2 d\xi d\tau \right)^{p/2} dx \frac{dt}{t} \right)^{1/p}$$

with the usual modification when  $p = \infty$ . We will refer to these spaces as *Z-spaces* starting from Subsection 5.1.3 (the letter  $L$  already being overused), with an indexing convention such that  $Z_\theta^p = L(p, \theta + 1, 2)$ . The boundary condition for  $(D_B)_{\theta,A}^p$  is phrased in terms of the trace operator, which is shown to be bounded from  $\dot{W}(p, \theta, 2)$  (the space of functions whose gradients are in  $L(p, \theta, 2)$ ) to  $\dot{B}_\theta^{p,p}$  when  $p > n/(n + \theta)$  [21, Theorem 3.9]. A similar argument is used to define the boundary conormal derivative  $\partial_{\nu_A} u|_{\partial\mathbb{R}_+^{1+n}}$ .

As we stated earlier, it will be technically convenient for us to consider Regularity problems rather than Dirichlet problems. We would also prefer to stick with problems of order between  $-1$  and  $0$ . To this end we define, for  $-1 < \theta < 0$  and  $p$  such that  $n/(n + \theta + 1) < p \leq \infty$ ,

$$(R_B)_{\theta,A}^p : \begin{cases} L_A u = 0 & \text{in } \mathbb{R}_+^{1+n}, \\ \lim_{t \rightarrow 0} \nabla_{\parallel} u(t, \cdot) = \nabla_{\parallel} f \in \dot{B}_\theta^{p,p}(\mathbb{R}^n : \mathbb{C}^{mn}) \\ \lim_{t \rightarrow \infty} \nabla_{\parallel} u(t, \cdot) = 0 & \text{in } \mathcal{Z}'(\mathbb{R}^n : \mathbb{C}^{mn}) \\ \|\nabla u\|_{Z_\theta^p} \lesssim \|\nabla_{\parallel} f\|_{\dot{B}_\theta^{p,p}} \end{cases}$$

and

$$(N_B)_{\theta,A}^p : \begin{cases} L_A u = 0 & \text{in } \mathbb{R}_+^{1+n}, \\ \lim_{t \rightarrow 0} \partial_{\nu_A} u(t, \cdot) = \partial_{\nu_A} f \in \dot{B}_\theta^{p,p}(\mathbb{R}^n : \mathbb{C}^m) \\ \lim_{t \rightarrow \infty} \nabla_{\parallel} u(t, \cdot) = 0 & \text{in } \mathcal{Z}'(\mathbb{R}^n : \mathbb{C}^{mn}) \\ \|\nabla u\|_{Z_\theta^p} \lesssim \|\partial_{\nu_A} f\|_{\dot{B}_\theta^{p,p}}, \end{cases}$$

replacing the trace conditions with limiting conditions for consistency with the ‘endpoint order’ problems that we have already defined,<sup>4</sup> writing  $Z_\theta^p$  instead of  $L(p, \theta+1, 2)$ , and including a decay condition at infinity. When  $p = \infty$  we impose the boundary condition in the weak-star topology. If we omit the decay condition at infinity, the Regularity problem  $(R_B)_{\theta,A}^p$  is equivalent to the Dirichlet problem  $(D_B)_{\theta+1,A}^p$  defined above by an argument similar to that of Remark 4.1.1, and the Neumann problem  $(N_B)_{\theta,A}^p$  is simply a rewriting of the previously-defined Neumann problem.

The Besov spaces  $\dot{B}_\theta^{p,p}$  with  $\theta \in (-1, 0)$  are not the only function spaces situated between  $H_0^p$  and  $\dot{H}_{-1}^p$ . One can also consider the *Hardy–Sobolev* spaces  $\dot{H}_\theta^p$  with  $\theta \in (-1, 0)$ . These are defined in Subsection 5.1.5; they may be identified with the homogeneous Triebel–Lizorkin spaces  $\dot{F}_\theta^{p,2}$ , whereas the Besov spaces  $\dot{B}_\theta^{p,p}$  may be identified with  $\dot{F}_\theta^{p,p}$  when  $p < \infty$ . We use Hardy–Sobolev spaces to formulate the following Regularity and Neumann problems, with  $-1 < \theta < 0$  and  $n/(n + \theta + 1) < p < \infty$ ,

$$(R_H)_{\theta,A}^p : \begin{cases} L_A u = 0 & \text{in } \mathbb{R}_+^{1+n}, \\ \lim_{t \rightarrow 0} \nabla_{\parallel} u(t, \cdot) = \nabla_{\parallel} f \in \dot{H}_\theta^p(\mathbb{R}^n : \mathbb{C}^{mn}) \\ \lim_{t \rightarrow \infty} \nabla_{\parallel} u(t, \cdot) = 0 & \text{in } \mathcal{Z}'(\mathbb{R}^n : \mathbb{C}^{mn}) \\ \|\nabla u\|_{T_\theta^p} \lesssim \|\nabla_{\parallel} f\|_{\dot{H}_\theta^p} \end{cases}$$

and

$$(N_H)_{\theta,A}^p : \begin{cases} L_A u = 0 & \text{in } \mathbb{R}_+^{1+n}, \\ \lim_{t \rightarrow 0} \partial_{\nu_A} u(t, \cdot) = \partial_{\nu_A} f \in \dot{H}_\theta^p(\mathbb{R}^n : \mathbb{C}^m) \\ \lim_{t \rightarrow \infty} \nabla_{\parallel} u(t, \cdot) = 0 & \text{in } \mathcal{Z}'(\mathbb{R}^n : \mathbb{C}^{mn}) \\ \|\nabla u\|_{T_\theta^p} \lesssim \|\partial_{\nu_A} f\|_{\dot{H}_\theta^p}. \end{cases}$$

Furthermore, for  $-1 < \theta < 0$  we formulate ‘endpoint’ problems  $(R_H)_{\theta,A}^\infty$  and  $(N_H)_{\theta,A}^\infty$  by replacing  $\dot{H}_\theta^p$  with the homogeneous BMO-Sobolev space  $\dot{\text{BMO}}_\theta$ , which may be identified with the homogeneous Triebel–Lizorkin spaces  $\dot{F}_\theta^{\infty,2}$ . In this case the boundary condition is imposed in the weak-star topology. In contrast with the boundary value problems with Besov space data, in these cases there is no trace theorem for the function space defined by  $\nabla u \in T_\theta^p$  (this is also the case for  $\theta = -1$  and  $\theta = 0$  with the spaces defined by  $\nabla u \in T_{-1}^p$  and  $\tilde{N}_*(\nabla u) \in L^p$  respectively).

Let us briefly summarise the Regularity and Neumann problems that we have introduced. At order zero we have problems  $(R_H)_{0,A}^p$  and  $(N_H)_{0,A}^p$ , which have

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<sup>4</sup>The trace conditions may be removed by invoking [21, Theorem 6.3], the trace theorem for functions with gradients in  $Z_\theta^p$ .

boundary data in  $H^p$  and a modified non-tangential maximal estimate on the interior. At order  $-1$  we have  $(R_H)_{-1,A}^p$  and  $(N_H)_{-1,A}^p$ , with boundary data in  $\dot{H}_{-1}^p$  and a  $T_{-1}^p$  interior estimate, and also  $(R_H)_{-1,A}^{(\infty,\alpha)}$  and  $(N_H)_{-1,A}^{(\infty,\alpha)}$  with boundary data in  $\dot{\Lambda}_{\alpha-1}$  (or  $\dot{\text{BMO}}_{-1}$  when  $\alpha = 0$ ) and a  $T_{-1;\alpha}^\infty$  interior estimate. In between, i.e. for order  $\theta \in (-1, 0)$ , we have  $(R_B)_{\theta,A}^p$  and  $(N_B)_{\theta,A}^p$  with boundary data in  $\dot{B}_{\theta}^{p,p}$ , and  $(R_H)_{\theta,A}^p$  and  $(N_H)_{\theta,A}^p$  with boundary data in  $\dot{H}_{\theta}^p$ . In these cases the interior estimates are in  $Z_{\theta}^p$  and  $T_{\theta}^p$  respectively. For all problems of negative order we also impose a decay condition on  $\nabla_{\parallel} u(t, \cdot)$  as  $t \rightarrow \infty$  in the space  $\mathcal{Z}'$  of tempered distributions modulo polynomials. In many cases this decay condition is redundant (see Lemma 7.2.1).

Note that for  $p = 2$  and for all  $s$ , the problems  $(R_H)_{s,A}^2$  and  $(R_B)_{s,A}^2$  (and likewise for Neumann problems) coincide, since  $\dot{H}_s^2 = \dot{B}_s^{2,2}$  and  $Z_s^2 = T_s^2$ .

#### 4.1.2 The first-order approach: perturbed Dirac operators and Cauchy–Riemann systems

Let  $D$  denote the differential operator on  $\mathbb{C}^{m(1+n)}$ -valued functions given by

$$D := \begin{bmatrix} 0 & \text{div}_{\parallel} \\ -\nabla_{\parallel} & 0 \end{bmatrix}$$

with respect to the transversal/tangential splitting (4.2) of  $\mathbb{C}^{m(1+n)}$ . We refer to  $D$  as a *Dirac operator*, because  $D^2$  acts as the tangential Laplacian  $\Delta_{\parallel}$  on transversal functions. Suppose that  $B \in L^\infty(\mathbb{R}^n : \mathcal{L}(\mathbb{C}^{m(1+n)}))$  is a bounded coefficient matrix satisfying the same assumptions as those we previously assumed on  $A$ : boundedness, measurability, complexity,  $t$ -independence, and strict accretivity on curl-free vector fields. We refer to the operator  $DB$  as a *perturbed Dirac operator*.

The *Cauchy–Riemann system* associated with  $DB$  is the first-order partial differential system

$$(\text{CR})_{DB} : \begin{cases} \partial_t F + DBF = 0 & \text{in } \mathbb{R}_+^{1+n}, \\ \text{curl}_{\parallel} F_{\parallel} = 0 & \text{in } \mathbb{R}_+^{1+n} \end{cases} \quad (4.6)$$

interpreted in the weak ( $L_{\text{loc}}^2$ ) sense: that is, we say that  $F \in L_{\text{loc}}^2(\mathbb{R}_+^{1+n} : \mathbb{C}^{m(1+n)})$  solves  $(\text{CR})_{DB}$  if for all test functions  $\varphi \in C_c^\infty(\mathbb{R}_+^{1+n} : \mathbb{C}^{m(1+n)})$ ,

$$\iint_{\mathbb{R}_+^{1+n}} (F(t, x), \partial_t \varphi(t, x)) \, dx \, dt = \iint_{\mathbb{R}_+^{1+n}} (F(t, x), B^*(x) D \varphi(t, x)) \, dx \, dt$$

and for all  $\psi \in C_c^\infty(\mathbb{R}_+^{1+n} : \mathbb{C}^m)$  and  $1 \leq j, k \leq n$ ,  $j \neq k$ ,

$$\iint_{\mathbb{R}_+^{1+n}} (F_k(t, x), \partial_j \psi(t, x)) \, dx \, dt = - \iint_{\mathbb{R}_+^{1+n}} (F_j(t, x), \partial_k \psi(t, x)) \, dx \, dt.$$

The condition  $\operatorname{curl}_{\parallel} F_{\parallel} = 0$  is equivalent to the condition  $F \in \mathcal{R}(D)$ , the range of  $D$  (considered as acting on  $\mathbb{C}^{m(1+n)}$ -valued distributions modulo polynomials), and so the Cauchy–Riemann system  $(\operatorname{CR})_{DB}$  may be considered as an evolution equation in the space  $\mathcal{R}(D)$ .

The first-order approach to boundary value problems for elliptic systems  $L_A u = 0$  exploits a correspondence between these elliptic systems and Cauchy–Riemann systems  $(\operatorname{CR})_{DB}$ . Recall that  $A \in L^\infty(\mathbb{R}^n : \mathcal{L}(\mathbb{C}^{m(1+n)}))$ . Write  $A$  in matrix form with respect to the transversal/tangential splitting (4.2) of  $\mathbb{C}^{m(1+n)}$  as

$$A = \begin{bmatrix} A_{\perp\perp} & A_{\perp\parallel} \\ A_{\parallel\perp} & A_{\parallel\parallel} \end{bmatrix}, \quad (4.7)$$

and using this representation of  $A$  define auxiliary matrices

$$\overline{A} := \begin{bmatrix} A_{\perp\perp} & A_{\perp\parallel} \\ 0 & I \end{bmatrix} \quad \text{and} \quad \underline{A} := \begin{bmatrix} I & 0 \\ A_{\parallel\perp} & A_{\parallel\parallel} \end{bmatrix}$$

in  $L^\infty(\mathbb{R}^n : \mathcal{L}(\mathbb{C}^{m(1+n)}))$ . Strict accretivity of  $A$  implies that  $A_{\perp\perp}$  is invertible in  $L^\infty(\mathbb{R}^n : \mathcal{L}(\mathbb{C}^m))$ , and so  $\overline{A}$  is invertible in  $L^\infty(\mathbb{R}^n : \mathcal{L}(\mathbb{C}^{m(1+n)}))$ . Thus we may define

$$\hat{A} := \underline{A} \overline{A}^{-1}.$$

The transformed coefficient matrix  $\hat{A}$  is bounded and strictly accretive on curl-free vector fields as in (4.3), and  $\hat{A} = A$  [8, Proposition 3.2].

The  $A$ -conormal gradient  $\nabla_A u$  of a function  $u : \mathbb{R}_+^{1+n} \rightarrow \mathbb{C}^m$  is defined by

$$\nabla_A u = \begin{bmatrix} \partial_{\nu_A} u \\ \nabla_{\parallel} u \end{bmatrix}, \quad (4.8)$$

where the  $A$ -conormal derivative  $\partial_{\nu_A}$  is defined in (4.5). Notice that the components of  $\nabla_A u$  are exactly the quantities appearing in the boundary conditions of the Regularity and Neumann problems. This explains our preference for Regularity problems over Dirichlet problems.

The following theorem, due to Auscher, Axelsson (Rosén), and McIntosh, provides a bridge between elliptic equations  $L_A u = 0$  and Cauchy–Riemann systems  $(\operatorname{CR})_{DB}$ . See [8, §3], [7, Proposition 4.1], [79, §2], and [15, Lemma 7.1] for proofs and discussions.

**Theorem 4.1.3** (Auscher–Axelsson–McIntosh). *Let  $A$  be as above, and let  $B = \hat{A}$ . If  $u$  solves  $L_A u = 0$ , then the conormal gradient  $\nabla_A u$  solves the Cauchy–Riemann system  $(\text{CR})_{DB}$ . Conversely, if  $F$  solves  $(\text{CR})_{DB}$ , then there exists a function  $u$ , unique up to an additive constant, such that  $L_A u = 0$  and  $F = \nabla_A u$ .*

Therefore in our consideration of elliptic systems we may focus on Cauchy–Riemann systems if they are more useful. The principal advantage of Cauchy–Riemann systems over elliptic equations is that the Cauchy equation  $\partial_t F + DBF = 0$  can be solved by semigroup methods. We will sketch how this is done, following Auscher, Axelsson, and McIntosh [8] and Auscher and Axelsson [7]. This approach is the foundation for the rest of the article.

Consider  $D$  as an unbounded operator on  $L^2 := L^2(\mathbb{R}^n : \mathbb{C}^{m(1+n)})$  with natural domain, and consider  $B$  as a multiplication operator on  $L^2$ . Then, still assuming strict accretivity of  $B$  on  $\overline{\mathcal{R}(D)}$ ,<sup>5</sup> the composition  $DB$  is bisectorial and has bounded  $H^\infty$  functional calculus on its range [8, Proposition 3.3 and Theorem 3.4].<sup>6</sup> This is a highly non-trivial fact: it is part of the framework developed by Axelsson, Keith, and McIntosh [19], which encompasses the solution of the Kato square root problem [9].

Using the direct sum decomposition

$$L^2 = \mathcal{N}(DB) \oplus \overline{\mathcal{R}(DB)}$$

which follows from bisectoriality of  $DB$ , along with the bounded  $H^\infty$  functional calculus associated with  $DB$  on  $\overline{\mathcal{R}(DB)}$ , we obtain a decomposition

$$L^2 = \mathcal{N}(DB) \oplus \overline{\mathcal{R}(DB)}^+ \oplus \overline{\mathcal{R}(DB)}^-.$$

The *positive and negative spectral subspaces*  $\overline{\mathcal{R}(DB)}^\pm$  are the images of  $\overline{\mathcal{R}(DB)}$  under the projections  $\chi^\pm(DB)$ , which are defined via the functions  $\chi^\pm : \mathbb{C} \setminus i\mathbb{R} \rightarrow \{-1, 1\}$  given by

$$\chi^\pm(z) := \mathbf{1}_{z : \pm \operatorname{Re}(z) > 0};$$

$\chi^+$  and  $\chi^-$  are the characteristic functions of the right and left half-plane respectively. They are bounded and holomorphic on every bisector, so they fall within the scope of the  $H^\infty$  functional calculus.

On the positive spectral subspace  $\overline{\mathcal{R}(DB)}^+$  we can construct a strongly continuous semigroup  $(e^{-tDB})_{t>0}$  via the family of functions  $(z \mapsto e^{-tz})_{t>0}$ , which

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<sup>5</sup> $\overline{\mathcal{R}(D)}$  denotes the closure of the range of  $D$  in  $L^2(\mathbb{R}^n : \mathbb{C}^{m(1+n)})$ . We can obviously restrict attention to such functions when defining ‘strict accretivity on curl-free vector fields’.

<sup>6</sup>These notions are properly discussed in Section 5.2.1.



are holomorphic and bounded on the right half-plane. For each  $f \in \overline{\mathcal{R}(DB)}^+$  we may construct a *generalised Cauchy operator*  $C_{DB}^+ f$ , defined by

$$(C_{DB}^+ f)(t, x) := (e^{-tDB} f)(x).$$

The following theorem is a combination of parts of [8, Theorem 2.3] and [7, Corollary 8.4].

**Theorem 4.1.4** (Auscher–Axelsson–McIntosh). *If  $f \in \overline{\mathcal{R}(DB)}^+$ , then  $C_{DB}^+ f$  solves  $(CR)_{DB}$ , with*

$$\|\widetilde{N}_*(C_{DB}^+ f)\|_2 \simeq \|f\|_2 \quad \text{and} \quad \lim_{t \rightarrow 0} (C_{DB}^+ f)(t, \cdot) = f \quad \text{in } L^2.$$

*Conversely, if  $F$  solves  $(CR)_{DB}$  and  $\widetilde{N}_*(F) \in L^2$ , then  $F = C_{DB}^+ f$  for a unique  $f \in \overline{\mathcal{R}(DB)}^+$ .*

By combining this with Theorem 4.1.3, we obtain a new characterisation of well-posedness of the boundary value problems  $(R_H)_{0,A}^2$  and  $(N_H)_{0,A}^2$ . Consider the  $H^2$ -Regularity problem  $(R_H)_{0,A}^2$  and let  $B = \hat{A}$ . A function  $u$  solves  $L_A u = 0$  with  $\widetilde{N}_*(\nabla u) \in L^2$  ( $\nabla u$  and  $\nabla_A u$  are interchangeable in this assumption) if and only if  $\nabla_A u = C_{DB}^+ g$  for some  $g \in \overline{\mathcal{R}(DB)}^+$ , and therefore  $\nabla_{\parallel} u(t, \cdot) = (C_{DB}^+ g)(t)_{\parallel}$  and  $\lim_{t \rightarrow 0} \nabla_{\parallel} u(t, \cdot) = g_{\parallel}$ . Hence  $(R_H)_{0,A}^2$  is well-posed if and only if  $g \mapsto g_{\parallel}$  is an isomorphism from  $\overline{\mathcal{R}(DB)}^+$  to  $L^2(\mathbb{R}^n : \mathbb{C}^{mn}) \cap \mathcal{N}(\text{curl}_{\parallel})$ . By the same argument,  $(N_H)_{0,A}^2$  is well-posed if and only if  $g \mapsto g_{\perp}$  is an isomorphism from  $\overline{\mathcal{R}(DB)}^+$  to  $L^2(\mathbb{R}^n : \mathbb{C}^m)$ .

By characterising solutions to  $(CR)_{DB}$  within various function spaces, we can reduce well-posedness of corresponding Regularity and Neumann problems to proving that the transversal and tangential restriction maps are isomorphisms between certain function spaces ‘on the boundary’. However, in this section we only described how to handle boundary value problems of order 0 with  $L^2$  boundary data. We shall extend this technique to boundary value problems of more general order, and beyond  $L^2$ .

### Adapted function spaces

‘Adapted’ Hardy spaces  $H_L^p$ , with respect to which some operator  $L$  has good properties (such as bounded  $H^\infty$  functional calculus), have been developed in various contexts. For example, Hardy spaces of differential forms on Riemannian manifolds were constructed by Auscher, McIntosh, and Russ [13] (these are adapted to the Hodge-Dirac operator  $d + d^*$  on the de Rham complex); Hardy

spaces adapted to non-negative self-adjoint operators satisfying Davies–Gaffney estimates on spaces of homogeneous type were studied by Hofmann, Lu, Mitrea, Mitrea, and Yan [48] (generalising the work of Auscher, McIntosh, and Russ); Hardy spaces adapted to divergence-form elliptic operators on  $\mathbb{R}^n$  were developed by Hofmann and Mayboroda [49] and also McIntosh [50]. This is a very small sample of the work that has been done.

Hardy spaces  $\mathbf{H}_{DB}^p$  and Sobolev spaces  $\mathbf{W}_{-1,DB}^p$  adapted to perturbed Dirac operators  $DB$  were introduced by Auscher and Stahlhut [16] (see also Stahlhut’s thesis [84]).<sup>7</sup> These spaces consist of  $\mathbb{C}^{m(1+n)}$ -valued functions (at least formally); the simplest case is

$$\mathbf{H}_{DB}^2 = \overline{\mathcal{R}(DB)} \subset L^2 = L^2(\mathbb{R}^n : \mathbb{C}^{m(1+n)}).$$

The bounded  $H^\infty$  calculus of  $DB$  on  $\mathbf{H}_{DB}^2$  extends by boundedness to  $\mathbf{H}_{DB}^p$  and  $\mathbf{W}_{-1,DB}^p$ , yielding spectral decompositions

$$\mathbf{H}_{DB}^p = \mathbf{H}_{DB}^{p,+} \oplus \mathbf{H}_{DB}^{p,-}, \quad \mathbf{W}_{-1,DB}^p = \mathbf{W}_{-1,DB}^{p,+} \oplus \mathbf{W}_{-1,DB}^{p,-}.$$

Furthermore, the Cauchy operator  $C_{DB}^+$  on  $\overline{\mathcal{R}(DB)}^+$  extends to operators on  $\mathbf{H}_{DB}^{p,+}$  and  $\mathbf{W}_{-1,DB}^{p,+}$ , both of which we denote by  $\mathbf{C}_{DB}^+$ .

The main application of these spaces, which incorporates results from both [16] and the subsequent work of Auscher and Mourgoglou [15], is a classification of solutions to the Cauchy–Riemann system  $(\text{CR})_{DB}$  with various  $L^p$ -type interior estimates, for  $p$  such that certain  $DB$ -adapted spaces may be identified with  $D$ -adapted spaces.<sup>8</sup>

**Theorem 4.1.5** (Auscher–Mourgoglou–Stahlhut). *Let  $1 < p < \infty$  be such that  $\mathbf{H}_{DB}^p \simeq \mathbf{H}_D^p$ .*

(i) *If  $f \in \mathbf{H}_{DB}^{p,+}$ , then  $\mathbf{C}_{DB}^+ f$  solves  $(\text{CR})_{DB}$ , with*

$$\left\| \widetilde{N}_*(\mathbf{C}_{DB}^+ f) \right\|_p \simeq \|f\|_{H^p} \quad \text{and} \quad \lim_{t \rightarrow 0} \mathbf{C}_{DB}^+ f(t) = f \quad \text{in } H^p.$$

*Conversely, if  $F$  solves  $(\text{CR})_{DB}$  and  $\widetilde{N}_* F \in L^p$ , then  $F = \mathbf{C}_{DB}^+ f$  for some  $f \in \mathbf{H}_{DB}^{p,+}$ .*

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<sup>7</sup>We will not define  $\mathbf{H}_{DB}^p$  here, but only mention that it is defined, along with more general spaces, in Section 6.1.

<sup>8</sup>For simplicity we only state results for  $1 < p < \infty$  here. Corresponding results for  $p \leq 1$  and  $p = \infty$  (BMO and Hölder spaces) are also available.

(ii) If  $f \in \mathbf{W}_{-1,DB}^{p,+}$ , then  $\mathbf{C}_{DB}^+ f$  solves  $(\text{CR})_{DB}$ , with

$$\|\mathbf{C}_{DB}^+ f\|_{T_{-1}^p} \simeq \|f\|_{\dot{W}_{-1}^p} \quad \text{and} \quad \lim_{t \rightarrow 0} \mathbf{C}_{DB}^+ f(t) = f \quad \text{in } \dot{W}_{-1}^p.$$

Conversely, if  $F \in T_{-1}^p$  solves  $(\text{CR})_{DB}$  and  $\lim_{t \rightarrow \infty} F(t)_{\parallel} = 0$  in  $\mathcal{Z}'(\mathbb{R}^n)$ , then  $F = \mathbf{C}_{DB}^+ f$  for some  $f \in \mathbf{W}_{-1,DB}^{p,+}$ .

Furthermore, Auscher and Stahlhut [16, Theorem 5.1] show that for every  $B$  there exists an open interval  $I_0(\mathbf{H}, DB) \ni 2$  such that  $\mathbf{H}_{DB}^p \simeq \mathbf{H}_D^p$  for all  $p \in I_0(\mathbf{H}, DB)$ , thus yielding a nontrivial range of exponents for which Theorem 4.1.5 applies.

As we described in the  $p = 2$  case, Theorem 4.1.5 implies a characterisation of well-posedness of various Regularity and Neumann problems, both of order 0 and order  $-1$ , in terms of certain transversal and tangential restriction maps being isomorphisms. We will state our extension of this result in Theorem 4.1.7.

**The main goal of this article is to extend Theorem 4.1.5 to order  $s \in (-1, 0)$ , incorporating both Hardy–Sobolev spaces and Besov spaces.** To this end, we introduce Hardy–Sobolev spaces  $\mathbf{H}_{s,L}^p$  and Besov spaces  $\mathbf{B}_{s,L}^p$  adapted to operators  $L$  satisfying ‘standard assumptions’, which are satisfied in particular by the perturbed Dirac operators  $DB$  and  $BD$ . We define extension operators

$$(\mathbb{Q}_{\varphi,L} f)(t) = \varphi(tL)f \quad (t > 0, f \in \overline{\mathcal{R}(L)})$$

for appropriate holomorphic functions  $\varphi$ , and the adapted Hardy–Sobolev and Besov norms are then, roughly speaking, defined by

$$\|f\|_{\mathbf{H}_{s,L}^p} := \|\mathbb{Q}_{\varphi,L} f\|_{T_s^p}, \quad \|f\|_{\mathbf{B}_{s,L}^p} := \|\mathbb{Q}_{\varphi,L} f\|_{Z_s^p}.$$

These definitions are reminiscent of the  $\varphi$ -transform characterisations of Triebel–Lizorkin and Besov spaces due to Frazier and Jawerth [38], with functional calculus and tent/ $Z$ -spaces taking the place of discretised Littlewood–Paley decompositions and sequence spaces.

Chapters 5 and 6 are occupied with setting up a sufficiently rich general theory of adapted Hardy–Sobolev and Besov spaces. The theory is relatively straightforward once enough preliminaries have been collected, but this takes some time. We point out in particular the amount of work needed to establish independence on  $\varphi$  of the spaces  $\mathbf{H}_{s,L}^p$  and  $\mathbf{B}_{s,L}^p$  (essentially all of Sections 5.2.3 and 5.2.4) and the care which must be taken in discussing completions (Subsection 6.1.3), which is necessary to discuss interpolation.

### 4.1.3 Characterisation of solutions to CR systems, and applications to well-posedness

The main theorem of this article is the following classification of solutions to the Cauchy–Riemann system  $(\text{CR})_{DB}$ . In this statement we restrict ourselves to  $1 < p < \infty$ . Our theorem allows for  $p \leq 1$  and  $p = \infty$ , but the corresponding results are better stated in terms of the ‘exponent notation’ that we introduce in Subsection 5.1.1. See Theorems 7.3.1 and 7.3.2 for the full statements of this result.

**Theorem 4.1.6.** *Let  $-1 < s < 0$  and  $1 < p < \infty$ .*

(i) *Suppose that  $\mathbf{H}_{s,DB}^p = \mathbf{H}_{s,D}^p$ . If  $f \in \mathbf{H}_{s,DB}^{p,+}$ , then  $\mathbf{C}_{DB}^+ f$  solves  $(\text{CR})_{DB}$ , with*

$$\left\| \mathbf{C}_{DB}^+ f \right\|_{T_s^p} \simeq \|f\|_{\dot{H}_s^p} \quad \text{and} \quad \lim_{t \rightarrow 0} \mathbf{C}_{DB}^+ f(t) = f \quad \text{in } \dot{H}_s^p,$$

*and furthermore  $\lim_{t \rightarrow \infty} \mathbf{C}_{DB}^+ f(t)_{\parallel} = 0$  in  $\mathcal{Z}'(\mathbb{R}^n)$ . Conversely, if  $F \in T_s^p$  solves  $(\text{CR})_{DB}$  and  $\lim_{t \rightarrow \infty} F(t)_{\parallel} = 0$  in  $\mathcal{Z}'(\mathbb{R}^n)$ , then  $F = \mathbf{C}_{DB}^+ f$  for some  $f \in \mathbf{H}_{s,DB}^{p,+}$ .*

(ii) *Suppose that  $\mathbf{B}_{s,DB}^p = \mathbf{B}_{s,D}^p$ . If  $f \in \mathbf{B}_{s,DB}^{p,+}$ , then  $\mathbf{C}_{DB}^+ f$  solves  $(\text{CR})_{DB}$ , with*

$$\left\| \mathbf{C}_{DB}^+ f \right\|_{Z_s^p} \simeq \|f\|_{\dot{B}_s^{p,p}} \quad \text{and} \quad \lim_{t \rightarrow 0} \mathbf{C}_{DB}^+ f(t) = f \quad \text{in } \dot{B}_s^{p,p}.$$

*and furthermore  $\lim_{t \rightarrow \infty} \mathbf{C}_{DB}^+ f(t)_{\parallel} = 0$  in  $\mathcal{Z}'(\mathbb{R}^n)$ . Conversely, if  $F \in Z_s^p$  solves  $(\text{CR})_{DB}$  and  $\lim_{t \rightarrow \infty} F(t)_{\parallel} = 0$  in  $\mathcal{Z}'(\mathbb{R}^n)$ , then  $F = \mathbf{C}_{DB}^+ f$  for some  $f \in \mathbf{B}_{s,DB}^{p,+}$ .*

Parts (i) and (ii) of this theorem are essentially identical, the only modifications being the replacement of (adapted) Hardy–Sobolev spaces with (adapted) Besov spaces, and of tent spaces with  $Z$ -spaces. In fact, our arguments apply equally to both parts, and we prove them simultaneously. Although the theorem can be thought of as ‘intermediate to’ Theorem 4.1.5, it does not simply follow by any interpolation procedure. It is proven similarly, but the underlying techniques must be generalised, and this takes a considerable amount of work. Neither direction is easy, but the ‘converse’ direction is certainly the harder one.

Starting from information on the intervals  $I_0(\mathbf{H}, DB), I_0(\mathbf{H}, DB^*) \ni 2$  (as given by Auscher and Stahlhut), a procedure of ‘ $\heartsuit$ -duality’ and interpolation allows us to find non-trivial regions  $I(\mathbf{H}, DB)$  and  $I(\mathbf{B}, DB)$  of exponents  $(p, s)$  for which Theorem 4.1.6 applies (this is done in Subsection 6.2.1).

With Theorem 4.1.6 as a springboard, we are able to extend the characterisation of well-posedness of Regularity and Neumann problems, described for  $p = 2$  after the statement of Theorem 4.1.3 and then extended to  $p \neq 2$  and  $s \in \{-1, 0\}$  by Auscher, Mourougolou, and Stahlhut, as follows.

When  $-1 \leq s \leq 0$  and  $1 < p < \infty$  (and in fact for a slightly larger range of exponents),  $\mathbf{H}_{s,D}^p$  is equal to the set of those  $f \in \dot{H}_s^p(\mathbb{R}^n : \mathbb{C}^{m(1+n)})$  with  $\text{curl}_{\parallel} f_{\parallel} = 0$ . Let  $N_{\perp}$  and  $N_{\parallel}$  denote the projections from  $\mathbf{H}_{s,D}^p$  onto  $\mathbf{H}_{s,\perp}^p := \dot{H}_s^p(\mathbb{R}^n : \mathbb{C}^m)$  and  $\mathbf{H}_{s,\parallel}^p := \dot{H}_s^p(\mathbb{R}^n : \mathbb{C}^{mn}) \cap \mathcal{N}(\text{curl}_{\parallel})$  respectively. For  $(p, s)$  as in Theorem 4.1.6 we have an identification of  $\mathbf{H}_{s,DB}^{p,+}$  as a subset of  $\mathbf{H}_{s,D}^p$ , and so we can use the projections  $N_{\perp}$  and  $N_{\parallel}$  to define

$$N_{H,DB,\parallel}^{(p,s)} : \mathbf{H}_{s,DB}^{p,+} \rightarrow \mathbf{H}_{s,\parallel}^p \quad \text{and} \quad N_{H,DB,\perp}^{(p,s)} : \mathbf{H}_{s,DB}^{p,+} \rightarrow \mathbf{H}_{s,\perp}^p$$

Corresponding definitions of  $N_{B,DB,\parallel}^{(p,s)}$  and  $N_{B,DB,\perp}^{(p,s)}$  are also made for Besov spaces. It is these operators that carry the well-posedness of Regularity and Neumann problems, as shown by the following theorem.<sup>9</sup> The  $s \in \{-1, 0\}$  endpoints follow from Theorem 4.1.5.

**Theorem 4.1.7.** *Let  $B = \hat{A}$ ,  $-1 \leq s \leq 0$ , and  $1 < p < \infty$ . Suppose that  $\mathbf{H}_{s,DB}^p = \mathbf{H}_{s,D}^p$ . Then  $(R_H)_{s,A}^p$  (resp.  $(N_H)_{s,A}^p$ ) is well-posed if and only if  $N_{H,DB,\parallel}^{(p,s)}$  (resp.  $N_{H,DB,\perp}^{(p,s)}$ ) is an isomorphism. The same results hold mutatis mutandi for Besov spaces.*

For all coefficients  $A$ , the Lax–Milgram theorem guarantees well-posedness of the problems  $(R_H)_{-1/2,A}^2$  and  $(N_H)_{-1/2,A}^2$  (see [12, Theorems 3.2 and 3.3]). We refer to solutions of these boundary value problems as *energy solutions*. There are certain situations where  $(R_H)_{s,A}^p$  is well-posed for some  $(p, s)$ , but where the unique solution to  $(R_H)_{s,A}^p$  with boundary data  $f \in \dot{H}_{-1/2}^2 \cap \dot{H}_s^p$  (*energy data*) is not the corresponding energy solution. This is shown in [18] for the Dirichlet problems. This behaviour shows why we insist on specifying an interior estimate in the definitions of our boundary value problems.

We say that a boundary value problem (as above) is *compatibly well-posed* if it is well-posed, and if in addition the unique solution to the boundary value problem with energy data is the energy solution. By Theorem 4.1.7,  $(R_H)_{s,A}^p$  is compatibly well-posed if  $N_{H,DB,\parallel}^{(p,s)}$  is an isomorphism, and if the inverses  $(N_{H,DB,\parallel}^{(p,s)})^{-1}$  and  $(N_{H,DB,\perp}^{(2,-1/2)})^{-1}$  are consistent, in the sense that they are equal on the intersection

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<sup>9</sup>The full theorem (Theorem 7.4.4) allows for  $p \leq 1$  and  $p = \infty$  (and again, uses new ‘exponent notation’).

$\mathbf{H}_{s,\parallel}^p \cap \mathbf{H}_{-1/2,\parallel}^2$  (and likewise for Neumann problems, and with Besov spaces). This allows us to interpolate compatible well-posedness as a straightforward corollary of Theorem 4.1.7.<sup>10</sup> Furthermore, by using real interpolation, we can deduce compatible well-posedness of boundary value problems with Besov boundary data from that of those with Hardy–Sobolev boundary data.

**Theorem 4.1.8.** *Suppose  $-1 \leq s_0, s_1 \leq 0$ ,  $1 < p_0, p_1 < \infty$ , and  $\alpha \in (0, 1)$ , and let*

$$\frac{1}{p} = \frac{1-\alpha}{p_0} + \frac{\alpha}{p_1} \quad \text{and} \quad s = (1-\alpha)s_0 + \alpha s_1.$$

(i) *If  $\mathbf{H}_{s_j,DB}^{p_j} = \mathbf{H}_{s_0,D}^{p_j}$  for  $j = 0, 1$ , and if  $(R_H)_{s_0,A}^{p_0}$  and  $(R_H)_{s_1,A}^{p_1}$  are compatibly well-posed, then  $(R_H)_{s,A}^p$  is compatibly well-posed, and furthermore if  $s_0 \neq s_1$  then  $(R_B)_{s,A}^p$  is compatibly well-posed.*

(ii) *If  $\mathbf{B}_{s_j,DB}^{p_j} = \mathbf{B}_{s_0,D}^{p_j}$  for  $j = 0, 1$ , and if  $(R_B)_{s_0,A}^{p_0}$  and  $(R_B)_{s_1,A}^{p_1}$  are compatibly well-posed, then  $(R_B)_{s,A}^p$  is compatibly well-posed.*

*Corresponding results are also true for Neumann problems.*

Since invertibility is stable in complex interpolation scales, well-posedness of our boundary value problems is also stable, in the following sense.<sup>11</sup>

**Theorem 4.1.9.** *Let  $-1 < s < 0$  and  $1 < p < \infty$ , and suppose that  $\mathbf{H}_{s_0,DB}^{p_0} = \mathbf{H}_{s_0,D}^{p_0}$  for all  $(p_0, s_0)$  in some neighbourhood of  $(p, s)$  (in the usual topology on  $\mathbb{R}^2$ ). Suppose also that  $(R_H)_{s,A}^p$  is (compatibly) well-posed. Then  $(R_H)_{s_1,A}^{p_1}$  is (compatibly) well-posed for all  $(p_1, s_1)$  in some neighbourhood of  $(p, s)$ . Similar results hold for Neumann problems and with Besov spaces.*

Note that well-posedness extrapolates to well-posedness, and compatible well-posedness extrapolates to compatible well-posedness.

Finally, we have the duality result for well-posedness.<sup>12</sup>

**Theorem 4.1.10.** *Let  $-1 \leq s \leq 0$  and  $1 < p < \infty$ . Then  $(R_H)_{s,A}^p$  is (compatibly) well-posed if and only if  $(R_H)_{-s-1,A^*}^{p'}$  is (compatibly) well-posed, and similar results hold for Neumann problems and with Besov spaces.*

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<sup>10</sup>As with the other theorems, we have not stated this in full generality. The full result is Theorem 7.4.5.

<sup>11</sup>The full result here is Theorem 7.4.6.

<sup>12</sup>See Theorem 7.4.8.

Note that the mapping  $(p, s) \mapsto (p', -s - 1)$  can be seen as a reflection about the point  $(1/2, -1/2)$  in the  $(1/p, s)$ -plane. This corresponds to what we will later refer to as ‘ $\heartsuit$ -duality’.

These theorems can be used to derive new well-posedness results for Regularity problems  $(R_H)_{s,A}^p$  with fractional order  $s \in (-1, 0)$ , and also to derive known results for  $(R_B)_{s,A}^p$  which were recently obtained by different methods by Barton and Mayboroda [21]. For details see Subsection 7.4.2.

## 4.2 Summary of the article

In Section 5.1 we introduce the various function spaces that we use, their basic properties, and their interrelations. There are two types of function spaces that we consider. First, the ‘ambient spaces’: tent spaces,  $Z$ -spaces, and slice spaces. Many of the results here are new, or have not been used in this context, so we make ourselves well acquainted with these spaces. The second type of space that we consider are the homogeneous ‘smoothness spaces’: Hardy–Sobolev spaces, Besov spaces, and so on. Since we do not establish any new properties of these spaces, we restrict ourselves to a quick review. We also introduce a new system of notation for exponents. This is not strictly necessary, but it greatly cleans up the exposition of later parts of the article and makes the flow of ideas more apparent.

In Section 5.2 we discuss the basic operator-theoretic notions that we will need. The operators that we use in applications (i.e. the perturbed Dirac operators  $DB$  and  $BD$ ) are bisectorial, with bounded  $H^\infty$  functional calculi on their ranges, and satisfying certain off-diagonal estimates. Most of the abstract theory we develop works for any operator  $A$  satisfying these ‘standard assumptions’, so we work with such operators until we are forced to use more specific properties of perturbed Dirac operators. We establish the boundedness of certain integral operators between tent spaces and  $Z$ -spaces. Particular examples of these operators are given in terms of ‘extension’ and ‘contraction’ operators  $\mathbb{Q}_{\varphi,A}$  and  $\mathbb{S}_{\psi,A}$ , which we will introduce and discuss. This section culminates in Theorem 5.2.20, which quantifies when operators of the form  $\mathbb{Q}_{\psi,A}\eta(A)\mathbb{S}_{\varphi,A}$  are bounded between different tent/ $Z$ -spaces, where  $\eta$  is a holomorphic function on an appropriate bisector which is not necessarily bounded.

In Section 6.1 we define and investigate Hardy–Sobolev and Besov spaces adapted to an operator  $A$  satisfying the aforementioned standard assumptions. First we introduce ‘pre’-Besov–Hardy–Sobolev spaces  $\mathbb{H}_A^{\mathbf{p}}$  and  $\mathbb{B}_A^{\mathbf{p}}$  and establish

their basic properties in Subsection 6.1.1. Mapping properties of the holomorphic functional calculus between these spaces, including boundedness for  $H^\infty$  functions of  $A$  and ‘regularity shifting’ estimates for operators such as powers of  $A$ , are collected in Subsection 6.1.2. These all follow from Theorem 5.2.20. In Subsection 6.1.3 we discuss completions. This is more subtle than it initially seems. We define ‘canonical completions’  $\psi\mathbf{H}_A^{\mathbf{p}}$  and  $\psi\mathbf{B}_A^{\mathbf{p}}$  in terms of an auxiliary functions  $\psi$ , and show how these can be used to formulate satisfactory duality and interpolation results (Proposition 6.1.19 and Theorem 6.1.23). Finally, in Subsection 6.1.4 we show that the Cauchy operators  $C_A^\pm$  produce strong solutions of the Cauchy–Riemann equation  $(\text{CR})_A$  with initial data in any completion of any pre-Besov–Hardy–Sobolev space, and we also show the quasi-norm equivalence

$$\|f\|_{\mathbb{H}_A^{\mathbf{p}}} \simeq \|C_A^\pm f\|_{T^{\mathbf{p}}} \quad (f \in \mathbb{H}_A^{\mathbf{p}, \pm}) \quad (4.9)$$

when  $\mathbf{p} = (p, s)$  with  $p \leq 2$  and  $s < 0$ , and likewise for Besov spaces and  $Z$ -spaces (Theorem 6.1.25). This is important because it implies that the Cauchy operators can be used to construct solutions of  $(\text{CR})_A$  which satisfy good tent/ $Z$ -space estimates, at least for this range of exponents  $\mathbf{p} = (p, s)$ .

Up until this point, we work with  $\mathbb{C}^N$ -valued functions for an arbitrary  $N \in \mathbb{N}$ , as in this abstract setting we gain nothing from the transversal/tangential structure of  $\mathbb{C}^{m(1+n)}$ .

In Section 6.2 we consider the case when  $A$  is a perturbed Dirac operator of the form  $DB$  or  $BD$  (and we finally specialise to  $\mathbb{C}^{m(1+n)}$ -valued functions). We show that for a large range of exponents  $\mathbf{p}$  the spaces  $\mathbf{H}_D^{\mathbf{p}}$  and  $\mathbf{B}_D^{\mathbf{p}}$  may be realised as projections of classical smoothness spaces (Theorem 6.2.1). Then we define ‘identification regions’  $I(\mathbf{H}, DB)$  and  $I(\mathbf{B}, DB)$ , consisting of exponents  $\mathbf{p}$  for which we can identify  $\mathbf{H}_D^{\mathbf{p}}$  and  $\mathbf{B}_D^{\mathbf{p}}$  as completions of  $\mathbb{H}_{DB}^{\mathbf{p}}$  and  $\mathbb{B}_{DB}^{\mathbf{p}}$  respectively. These regions turn out to be stable under interpolation and  $\heartsuit$ -duality (in a sense which interchanges  $B$  and  $B^*$ ). Finally, in Theorem 6.2.12 we show that for  $\mathbf{p} = (p, s) \in I(\mathbf{H}, DB)$  with  $s < 0$  we have boundedness of the Cauchy operator  $C_{DB}^+$  from  $\mathbb{H}_{DB}^{\mathbf{p}}$  to  $T^{\mathbf{p}}$ , extending the ‘abstract’ estimate (4.9) (and likewise for Besov spaces and  $Z$ -spaces). This is a long argument which requires various ad-hoc estimates. The result is known to fail for  $s = 0$ , so it does not follow by interpolation.

After presenting some basic properties of gradients of solutions of  $L_A u = 0$  (or equivalently solutions of  $(\text{CR})_{DB}$ ) we prove Theorems 7.3.1 and 7.3.2, the classification of solutions to  $(\text{CR})_{DB}$  in tent/ $Z$ -spaces with a decay condition at



infinity.<sup>13</sup> The argument is quite long, particularly for exponents  $\mathbf{p} = (p, s)$  with  $p > 2$ , and uses all the preceding material. We have been (perhaps excessively) pedantic in citing dependence on previous results, so it should be possible to treat certain technical lemmas as ‘black boxes’ in initial readings. We point out that although these results are ‘intermediate to’ the Auscher–Mourgoglou–Stahlhut theorem 4.1.5, and although it is proven with a similar argument, it does not follow by any interpolation procedure. The results must be reproven manually.<sup>14</sup>

In Section 7.4 we present straightforward (but still somewhat technical) applications to well-posedness and compatible well-posedness of Regularity and Neumann problems. These have already been summarised in the introduction (Subsection 4.1.3). In particular, we derive a range of well-posedness for the Regularity problem for real coefficient scalar equations in Subsection 7.4.2. For Hardy–Sobolev boundary data, this seems to be new. In Subsection 7.4.3 we state (without proof) a convergence result for Whitney averages of solutions to  $L_A u = 0$  within tent spaces and  $Z$ -spaces. Finally, we sketch the relationship between our approach and the method of layer potentials in Subsection 7.4.4. In the range of exponents  $\mathbf{p}$  for which our results hold, the solutions to boundary value problems are all given by (generalised) layer potentials.

## 4.3 Notation

The following notational conventions, some of them non-standard, will be used throughout the article.

For  $a, b \in \mathbb{R}$  and  $t > 0$  we write

$$m_a^b(t) := \begin{cases} t^a & (t \leq 1) \\ t^{-b} & (t \geq 1). \end{cases}$$

For  $0 < p, q \leq \infty$ , we define the number

$$\delta_{p,q} := \frac{1}{q} - \frac{1}{p},$$

with the interpretation  $1/\infty = 0$ .

We write the Euclidean distance on  $\mathbb{R}^n$  as  $d(x, y) = d(y, x) := |x - y|$ , the open ball with centre  $x \in \mathbb{R}^n$  and radius  $r > 0$  by  $B(x, r) := \{y \in \mathbb{R}^n : d(x, y) < r\}$ ,

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<sup>13</sup>The decay condition is removed for certain exponents in Section 7.2.

<sup>14</sup>Of course, we do manage to recycle some arguments from [16] and [15].

and the (half closed, half open) annulus with centre  $x \in \mathbb{R}^n$ , inner radius  $r_0 > 0$ , and outer radius  $r_1 > r_0$  by

$$A(x, r_0, r_1) := B(x, r_1) \setminus B(x, r_0) = \{y \in \mathbb{R}^n : r_0 \leq d(x, y) < r_1\}.$$

For subsets  $E, F \subset \mathbb{R}^n$ , we write

$$d(E, F) := \text{dist}(E, F) = \inf\{d(x, y) : x \in E, y \in F\}.$$

We let  $L^0(\Omega : E)$  denote the set of strongly measurable functions from a measure space  $\Omega$  to a Banach space  $E$ . For two quasi-Banach spaces  $X$  and  $Y$ , we write  $X \hookrightarrow Y$  to mean that  $X \subset Y$  (possibly after some identification has been made) and that the identity map is bounded. Often we will refer to norms as ‘quasinorms’ even though they are actually norms; for example, we will refer to the  $L^p$  quasinorm when  $p \in (0, \infty]$ , even though this is a norm when  $p \geq 1$ . For a quick introduction to quasi-Banach spaces the reader can consult the early sections of [56].

When necessary, we will label dual pairings by the space on the left: for example, by  $\langle f, g \rangle_{L^p}$ , we will mean the canonical duality pairing between  $L^p$  and  $L^{p'}$ , with  $f \in L^p$  and  $g \in L^{p'}$ .

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# Chapter 5

## Technical preliminaries

### 5.1 Function space preliminaries

Throughout this entire section we will consider  $\mathbb{C}^N$ -valued functions for some fixed  $N \in \mathbb{N}$ , but since nothing really changes whether we choose  $N = 1$  or  $N \neq 1$  (see Remark 5.1.13), we will not refer to  $\mathbb{C}^N$  in the notation. So we will write  $L^2(\mathbb{R}^n) = L^2(\mathbb{R}^n : \mathbb{C}^N)$ ,  $T^{\mathbf{p}}(\mathbb{R}^n) = T^{\mathbf{p}}(\mathbb{R}^n : \mathbb{C}^N)$ , and so on. For  $z \in \mathbb{C}^N$  we will write  $|z|$  in place of  $\|z\|_{\mathbb{C}^N}$ .

#### 5.1.1 Exponents

This work makes heavy use of the relationship between different exponents for function spaces. The most efficient way to do this, balancing economy of notation and clarity of ideas, is to introduce a new formalism for exponents right at the beginning, and work with it consistently.

Fix  $n \in \mathbb{N}_+$  corresponding to the dimension in which we will work. The following system of notation depends implicitly on  $n$ .

The set of *exponents* is the disjoint union

$$\mathbf{E} := \mathbf{E}_{\text{fin}} \sqcup \mathbf{E}_{\infty}$$

where  $\mathbf{E}_{\text{fin}} := \{(p, s) : p \in (0, \infty), s \in \mathbb{R}\}$  and  $\mathbf{E}_{\infty} := \{(\infty, s; \alpha) : s \in \mathbb{R}, \alpha \geq 0\}$ . We say that an exponent is *finite* if it is in  $\mathbf{E}_{\text{fin}}$ , and *infinite* if it is in  $\mathbf{E}_{\infty}$ .

We define two functions  $i : \mathbf{E} \rightarrow (0, \infty]$ ,  $r : \mathbf{E} \rightarrow \mathbb{R}$ , representing *integrability* and a kind of *regularity*, by

$$\begin{aligned} i(p, s) &:= p, & i(\infty, s; \alpha) &:= \infty, \\ r(p, s) &:= s, & r(\infty, s; \alpha) &:= s + \alpha. \end{aligned}$$

We also define functions  $j, \theta: \mathbf{E} \rightarrow \mathbb{R}$  by

$$\begin{aligned} j(p, s) &:= 1/p, & j(\infty, s; \alpha) &:= -\alpha/n \\ \theta(p, s) &:= s, & \theta(\infty, s; \alpha) &:= s. \end{aligned}$$

Note that  $\mathbf{p}$  is finite if and only if  $j(\mathbf{p})$  is positive, and furthermore every exponent  $\mathbf{p}$  is determined by the pair  $(j(\mathbf{p}), \theta(\mathbf{p}))$ .

For  $r \in \mathbb{R}$  and  $\mathbf{p} \in \mathbf{E}$ , define  $\mathbf{p} + r$  to be the unique exponent satisfying

$$j(\mathbf{p} + r) = j(\mathbf{p}) \quad \text{and} \quad \theta(\mathbf{p} + r) = \theta(\mathbf{p}) + r.$$

We similarly define  $\mathbf{p} - r$ .

For every exponent  $\mathbf{p}$ , we define the *dual exponent*  $\mathbf{p}'$  to be the unique exponent satisfying  $j(\mathbf{p}') + j(\mathbf{p}) = 1$  and  $\theta(\mathbf{p}') + \theta(\mathbf{p}) = 0$ . Concretely, for finite exponents we have

$$(p, s)' := \begin{cases} (p', -s) & (p > 1) \\ (\infty, -s; n(\frac{1}{p} - 1)) & (p \leq 1) \end{cases}$$

where  $p'$  is the usual Hölder conjugate of  $p$ . Clearly  $\mathbf{p}'' = \mathbf{p}$ . We also define the  $\heartsuit$ -dual exponent

$$\mathbf{p}^\heartsuit := \mathbf{p}' - 1,$$

and a quick computation shows that  $\mathbf{p}^{\heartsuit\heartsuit} = \mathbf{p}$ .

For two exponents  $\mathbf{p}, \mathbf{q} \in \mathbf{E}$ , we write  $\mathbf{p} \hookrightarrow \mathbf{q}$  to mean that

$$\theta(\mathbf{p}) \geq \theta(\mathbf{q}) \quad \text{and} \quad \theta(\mathbf{q}) - \theta(\mathbf{p}) = n(j(\mathbf{q}) - j(\mathbf{p})).$$

We always have  $\mathbf{p} \hookrightarrow \mathbf{p}$ . Observe that  $\mathbf{p} \hookrightarrow \mathbf{q}$  and  $\mathbf{q} \hookrightarrow \mathbf{r}$  implies  $\mathbf{p} \hookrightarrow \mathbf{r}$ , and  $\mathbf{p} \hookrightarrow \mathbf{q}$  if and only if  $\mathbf{q}' \hookrightarrow \mathbf{p}'$ . We define the Sobolev exponent  $\mathbf{p}^*$  be the unique exponent satisfying  $\mathbf{p} \hookrightarrow \mathbf{p}^*$  and  $\theta(\mathbf{p}^*) = \theta(\mathbf{p}) - 1$ .

For  $\eta \in \mathbb{R}$ , define  $[\mathbf{p}, \mathbf{q}]_\eta$  to be the unique exponent satisfying

$$\begin{aligned} j([\mathbf{p}, \mathbf{q}]_\eta) &= (1 - \eta)j(\mathbf{p}) + \eta j(\mathbf{q}), \\ \theta([\mathbf{p}, \mathbf{q}]_\eta) &= (1 - \eta)\theta(\mathbf{p}) + \eta\theta(\mathbf{q}). \end{aligned}$$

Note that  $[\mathbf{p}, \mathbf{q}]_0 = \mathbf{p}$  and  $[\mathbf{p}, \mathbf{q}]_1 = \mathbf{q}$ . Note also that  $\mathbf{p} \hookrightarrow \mathbf{q}$  if and only if  $\mathbf{q} = [\mathbf{p}, \mathbf{p}^*]_\eta$  for some  $\eta \geq 0$ .

**Lemma 5.1.1.** *Suppose  $\mathbf{p}$  and  $\mathbf{q}$  are exponents with  $\mathbf{p} \hookrightarrow \mathbf{q}$ . Then  $[\mathbf{p}, \mathbf{q}]_{\eta_0} \hookrightarrow [\mathbf{p}, \mathbf{q}]_{\eta_1}$  whenever  $\eta_0 \leq \eta_1$ .*

*Proof.* Write

$$\begin{aligned}
\theta([\mathbf{p}, \mathbf{q}]_{\eta_1}) - \theta([\mathbf{p}, \mathbf{q}]_{\eta_0}) &= ((1 - \eta_1)\theta(\mathbf{p}) + \eta_1\theta(\mathbf{q})) - ((1 - \eta_0)\theta(\mathbf{p}) + \eta_0\theta(\mathbf{q})) \\
&= (\eta_1 - \eta_0)(\theta(\mathbf{q}) - \theta(\mathbf{p})) \tag{5.1} \\
&= n(\eta_1 - \eta_0)(j(\mathbf{q}) - j(\mathbf{p})) \tag{5.2} \\
&= n(((1 - \eta_1)j(\mathbf{p}) + \eta_1j(\mathbf{q})) - ((1 - \eta_0)j(\mathbf{p}) + \eta_0j(\mathbf{q}))) \\
&= n(j([\mathbf{p}, \mathbf{q}]_{\eta_1}) - j([\mathbf{p}, \mathbf{q}]_{\eta_0})).
\end{aligned}$$

Where line (5.2) follows from  $\mathbf{p} \hookrightarrow \mathbf{q}$ . Furthermore, line (5.1),  $\eta_1 - \eta_0 \geq 0$ , and  $\theta(\mathbf{p}) \geq \theta(\mathbf{q})$  imply that  $\theta([\mathbf{p}, \mathbf{q}]_{\eta_0}) \geq \theta([\mathbf{p}, \mathbf{q}]_{\eta_1})$ . Thus  $[\mathbf{p}, \mathbf{q}]_{\eta_0} \hookrightarrow [\mathbf{p}, \mathbf{q}]_{\eta_1}$ .  $\square$

A straightforward computation shows the following lemma.

**Lemma 5.1.2.** *Suppose  $\mathbf{p} \hookrightarrow \mathbf{q}$  and  $\eta_0, \eta_1, \lambda \in \mathbb{R}$ . Then*

$$[[\mathbf{p}, \mathbf{q}]_{\eta_0}, [\mathbf{p}, \mathbf{q}]_{\eta_1}]_{\lambda} = [\mathbf{p}, \mathbf{q}]_{(1-\lambda)\eta_0 + \lambda\eta_1}.$$

*In particular this implies*

$$\begin{aligned}
\mathbf{p} &= [[\mathbf{p}, \mathbf{q}]_{-1}, \mathbf{q}]_{1/2} \\
\mathbf{q} &= [\mathbf{p}, [\mathbf{p}, \mathbf{q}]_2]_{1/2}.
\end{aligned}$$

The most convenient way of visualising exponents and relations between them is as points in the  $(j, \theta)$  plane. In Figure 5.1 we show two exponents  $\mathbf{p}$  and  $\mathbf{q}$  with  $\mathbf{p} \hookrightarrow \mathbf{q}$ , their dual exponents, their  $\heartsuit$ -duals, and various other exponents which may be constructed from them. The operations  $\mathbf{p} \mapsto \mathbf{p}'$  and  $\mathbf{p} \mapsto \mathbf{p}^\heartsuit$  are given by reflection about the marked points at  $(1/2, 0)$  and  $(1/2, -1/2)$  respectively.<sup>1</sup> Observe that we have  $\mathbf{p} \hookrightarrow \mathbf{q}$  if and only if the line segment from  $\mathbf{p}$  to  $\mathbf{q}$  is parallel the line from  $((n+1)/n, 0)$  to  $(1, -1)$  with the same orientation.

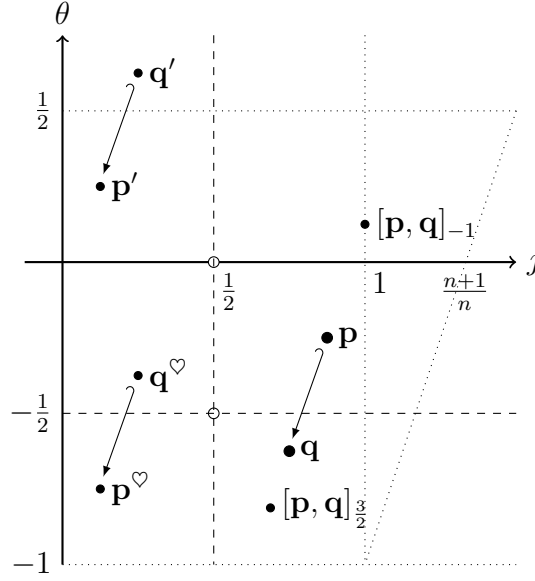
### 5.1.2 Tent spaces

The most fundamental function spaces in this work are the tent spaces. These were first introduced by Coifman, Meyer, and Stein [32, 33], and they have since proven their worth in harmonic analysis and PDE. The other ‘ambient spaces’ that we will use, namely  $Z$ -spaces and slice spaces, are closely related to tent spaces, so a solid knowledge of tent spaces will be useful.

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<sup>1</sup>The exponent  $(1/2, -1/2)$  is special: in Section 7.4 we introduce it as the ‘energy exponent’. Certain boundary value problems associated with this exponent are automatically well-posed due to the Lax–Milgram theorem.

Figure 5.1: Various exponents in the  $(j, \theta)$  plane.



For  $x \in \mathbb{R}^n$  we define the cone with vertex  $x$  by

$$\Gamma(x) := \{(t, y) \in \mathbb{R}_+^{1+n} : y \in B(x, t)\}$$

where  $B(x, t)$  is the open ball with centre  $x$  and radius  $t$ , and for each open ball  $B \subset X$  we define the *tent with base  $B$*  by

$$T(B) := \mathbb{R}_+^{1+n} \setminus \left( \bigcup_{x \notin B} \Gamma(x) \right).$$

Equivalently,  $T(B)$  is the set of points  $(y, t) \in \mathbb{R}_+^{1+n}$  such that  $B(y, t) \subset B$ .

The tent space quasinnorms are defined in terms of the *Lusin operator*  $\mathcal{A}$  and *Carleson operators*  $\mathcal{C}_\alpha$ . These are defined as follows. For all  $\alpha \geq 0$ ,  $f \in L^0(\mathbb{R}_+^{1+n})$  and  $x \in \mathbb{R}^n$ , we define

$$\mathcal{A}f(x) := \left( \iint_{\Gamma(x)} |f(t, y)|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2} \quad (5.3)$$

and

$$\mathcal{C}_\alpha f(x) := \sup_{B \ni x} \frac{1}{r_B^\alpha} \left( \frac{1}{r_B^n} \iint_{T(B)} |f(y, t)|^2 dy \frac{dt}{t} \right)^{1/2}.$$

For  $s \in \mathbb{R}$ , we define an operator  $\kappa^s$  on  $L^0(\mathbb{R}_+^{1+n})$  by

$$(\kappa^s f)(t, x) := t^s f(t, x)$$

for all  $(t, x) \in \mathbb{R}_+^{1+n}$ .

Now we will start using the exponent notation of Subsection 5.1.1, although it will not be truly useful just yet.

**Definition 5.1.3.** For a finite exponent  $\mathbf{p}$ , the *tent space*  $T^{\mathbf{p}} = T^{\mathbf{p}}(\mathbb{R}^n)$  is the set

$$T^{\mathbf{p}} = T_s^{\mathbf{p}} := \{f \in L^0(\mathbb{R}_+^{1+n}) : \mathcal{A}(\kappa^{-s}f) \in L^p(\mathbb{R}^n)\}$$

equipped with the quasinorm

$$\|f\|_{T_s^{\mathbf{p}}} := \left\| \mathcal{A}(\kappa^{-s}f) \right\|_{L^p(\mathbb{R}^n)}.$$

For an infinite exponent  $\mathbf{p} = (\infty, s; \alpha)$  we define  $T^{\mathbf{p}}$  by

$$T^{\mathbf{p}} = T_{s;\alpha}^{\infty} := \{f \in L^0(\mathbb{R}_+^{1+n}) : \mathcal{C}_{\alpha}(\kappa^{-s}f) \in L^{\infty}(\mathbb{R}^n)\}$$

with its natural norm.

*Remark 5.1.4.* The spaces  $T_s^{\mathbf{p}}$  agree with those defined by Hofmann, Mayboroda, and McIntosh [50, §8.3], and with the spaces  $T_{2,s}^{p,2}$  of Huang [51]. Our spaces  $T_{s;0}^{\infty}$  agree with Huang's spaces  $T_{2,s}^{\infty,2}$ .

All tent spaces are quasi-Banach spaces (Banach when  $i(\mathbf{p}) \geq 1$ ). For a finite exponent  $\mathbf{p}$  the subspace  $T^{\mathbf{p};c} \subset T^{\mathbf{p}}$  of compactly supported functions is dense in  $T^{\mathbf{p}}$ , and  $L_c^2(\mathbb{R}_+^{1+n})$  is densely contained in  $T^{\mathbf{p}}$ .

**Definition 5.1.5.** Let  $\mathbf{p}$  be an exponent with  $i(\mathbf{p}) \leq 1$ , and suppose  $B \subset \mathbb{R}^n$  is a ball. We say that a function  $a \in L^0(\mathbb{R}_+^{1+n})$  is a  $T^{\mathbf{p}}$ -atom (associated with  $B$ ) if  $a$  is essentially supported in  $T(B)$  and if

$$\|a\|_{T_s^2} \leq |B|^{\delta_{p,2}}.$$

where  $\delta_{p,2} = \frac{1}{2} - \frac{1}{p}$  (as defined in Section 4.3).

**Theorem 5.1.6** (Atomic decomposition). *Let  $\mathbf{p}$  be an exponent with  $i(\mathbf{p}) \leq 1$ . Then a function  $f \in L^0(\mathbb{R}_+^{1+n})$  is in  $T^{\mathbf{p}}$  if and only if there is a sequence  $(a_k)_{k \in \mathbb{N}}$  of  $T^{\mathbf{p}}$ -atoms and a sequence  $\lambda \in \ell^p(\mathbb{N})$  such that*

$$f = \sum_{k \in \mathbb{N}} \lambda_k a_k \tag{5.4}$$

with convergence in  $T^{\mathbf{p}}$ . Furthermore, we have

$$\|f\|_{T^{\mathbf{p}}} \simeq \inf \|\lambda\|_{\ell^p(\mathbb{N})}$$

where the infimum is taken over all such decompositions.

This is simply derived from the usual atomic decomposition theorem [33, Theorem 1c].

Note that the following duality theorem includes all finite exponents, without needing to separate the cases  $i(\mathbf{p}) \leq 1$  and  $i(\mathbf{p}) > 1$ . This is the first justification of our exponent notation.

**Theorem 5.1.7** (Duality). *Suppose that  $\mathbf{p}$  is a finite exponent. Then for all  $f, g \in L^0(\mathbb{R}_+^{1+n})$  we have*

$$\iint_{\mathbb{R}_+^{1+n}} |(f(t, x), g(t, x))| dx \frac{dt}{t} \lesssim \|f\|_{T^{\mathbf{p}}} \|g\|_{T^{\mathbf{p}'}} , \quad (5.5)$$

and the pairing

$$\langle f, g \rangle := \iint_{\mathbb{R}_+^{1+n}} (f(t, x), g(t, x)) dx \frac{dt}{t} \quad (5.6)$$

identifies the Banach space dual of  $T^{\mathbf{p}}$  with  $T^{\mathbf{p}'}$ .

Note in particular that the integral in (5.5) converges absolutely.

*Remark 5.1.8.* Throughout this article we will refer to the duality pairing appearing in (5.6) as the  $L^2$  duality pairing.

When  $\mathbf{p}$  is finite and  $i(\mathbf{p}) \geq 2$ ,  $T^{\mathbf{p}}$  may also be characterised in terms of the Carleson operator  $\mathcal{C}_0$ . This is a straightforward extension of [33, Theorem 3].

**Theorem 5.1.9** (Carleson characterisation of  $T^{\mathbf{p}}$ ). *Suppose  $\mathbf{p}$  is a finite exponent with  $i(\mathbf{p}) > 2$ . Then for all  $f \in L^0(\mathbb{R}_+^{1+n})$  we have*

$$\|f\|_{T^{\mathbf{p}}} \simeq \left\| \mathcal{C}_0(\kappa^{-\theta(\mathbf{p})} f) \right\|_{L^p(\mathbb{R}^n)} .$$

**Theorem 5.1.10** (Change of aperture). *For  $\beta \in (0, \infty)$  and  $x \in \mathbb{R}^n$  define*

$$\Gamma_\beta(x) := \{(t, y) \in \mathbb{R}_+^{1+n} : y \in B(x, \beta t)\},$$

and for  $f \in L^0(\mathbb{R}_+^{1+n})$  define  $\mathcal{A}_\beta f(x)$  as in (5.3), with  $\Gamma_\beta(x)$  in place of  $\Gamma(x)$ . Then for  $\beta \in (0, \infty)$  and each finite exponent  $\mathbf{p}$  we have an equivalence of quasinorms

$$\|f\|_{T^{\mathbf{p}}} \simeq \left\| \mathcal{A}_\beta(\kappa^{-\theta(\mathbf{p})} f) \right\|_{L^{i(\mathbf{p})}(\mathbb{R}^n)} .$$

This was proven by Coifman, Meyer, and Stein for  $T_0^p$  [33, Proposition 4] and Harboure, Torrea, and Viviani for  $T^{p,q}$  with  $q \in (1, \infty)$  [44, Proposition 2.3].<sup>2</sup> This can be simply extended to  $p, q \in (0, \infty)$  [3, Proposition 3.21], and

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<sup>2</sup>We have not defined the spaces  $T^{p,q}$  with  $q \neq 2$  here, because we will not use them.



the extension to the more general tent spaces here is immediate. Note that the method of proof in [3] requires knowledge of the result for  $q \neq 2$ .

The following embedding theorem, which can be seen as a tent space analogue of the Hardy–Littlewood–Sobolev embedding theorem, is proven in [4, Theorem 2.19].<sup>3</sup>

**Theorem 5.1.11** (Embeddings). *Let  $\mathbf{p}$  and  $\mathbf{q}$  be exponents with  $\mathbf{p} \hookrightarrow \mathbf{q}$ . Then we have the embedding*

$$T^{\mathbf{p}} \hookrightarrow T^{\mathbf{q}}.$$

The following complex interpolation theorem was proven by Hofmann, Mayboroda, and McIntosh for finite exponents [50, Lemma 8.23], and the extension to one infinite exponent follows by duality [4, Theorem 2.1].

**Theorem 5.1.12** (Complex interpolation). *Suppose  $\mathbf{p}$  and  $\mathbf{q}$  are exponents with  $j(\mathbf{p}), j(\mathbf{q}) \geq 0$  (with equality for at most one exponent), and  $0 < \theta < 1$ . Then we have the identification*

$$[T^{\mathbf{p}}, T^{\mathbf{q}}]_{\theta} = T^{[\mathbf{p}, \mathbf{q}]_{\theta}}.$$

*Remark 5.1.13.* In contrast with the article [4], we define the operator  $\kappa^s$  in terms of powers of  $t$  rather than powers of ball volumes, and so our tent spaces  $T_s^p(\mathbb{R}^n)$  correspond to the tent spaces  $T_{s/n}^{p,2}(\mathbb{R}^n)$  of [4]. We also use  $\mathbb{C}^N$ -valued functions instead of  $\mathbb{C}$ -valued functions. This does not change the validity of previous results, as one can always split  $T_s^p(\mathbb{R}^n : \mathbb{C}^N) \simeq \oplus_{j=1}^N T_s^p(\mathbb{R}^n : \mathbb{C})$  and apply the results to each summand individually. This reduction would fail if we were to replace  $\mathbb{C}^N$  with a general Banach space, but thankfully we have no need for such generality.

### 5.1.3 $Z$ -spaces

We now introduce a class of function spaces, called  $Z$ -spaces, which are related to tent spaces by real interpolation. The  $Z$ -spaces play the role for Besov spaces  $\dot{B}_s^{p,p}$  that the tent spaces play for Hardy–Sobolev spaces  $\dot{H}_s^p$ .

**Definition 5.1.14.** We refer to a pair

$$c = (c_0, c_1) \in (0, \infty) \times (3/2, \infty)$$

---

<sup>3</sup>The case where  $\mathbf{p}$  and  $\mathbf{q}$  are both infinite is not explicitly proven there, but it follows by the same argument.

as a *Whitney parameter*. To each Whitney parameter  $c$  and each  $(t, x) \in \mathbb{R}_+^{1+n}$  we associate the *Whitney region*

$$\Omega_c(t, x) := (c_1^{-1}t, c_1t) \times B(x, c_0t) \subset \mathbb{R}_+^{1+n},$$

and for  $f \in L^0(\mathbb{R}_+^{1+n})$  we define the  $L^2$ -*Whitney averages*

$$\mathcal{W}_c f(t, x) := \left( \iint_{\Omega_c(t, x)} |f(\tau, \xi)|^2 d\xi d\tau \right)^{1/2}.$$

For an exponent  $\mathbf{p}$  and a Whitney parameter  $c$ , and for all  $f \in L^0(\mathbb{R}_+^{1+n})$ , we define the quasinorm

$$\|f\|_{Z_c^{\mathbf{p}}} := \left\| \mathcal{W}_c(\kappa^{-r(\mathbf{p})} f) \right\|_{L^{i(\mathbf{p})}(\mathbb{R}_+^{1+n})} \quad (5.7)$$

(note the appearance of  $r(\mathbf{p})$  here) and a corresponding function space

$$Z_c^{\mathbf{p}} = Z_c^{\mathbf{p}}(\mathbb{R}^n) := \{f \in L^0(\mathbb{R}_+^{1+n}) : \|f\|_{Z_c^{\mathbf{p}}} < \infty\}.$$

For simplicity we write  $\Omega(t, x) := \Omega_{(1,2)}(t, x)$ .

*Remark 5.1.15.* The spaces  $Z_c^{\mathbf{p}}$  coincide with the spaces  $L(i(\mathbf{p}), r(\mathbf{p}) + 1, 2)$  introduced by Barton and Mayboroda [21]. In our applications these spaces will play the same role as they do in [21] - namely that of an ambient space for the gradient of a solution to an elliptic BVP with boundary data in a Besov space. The connection with tent spaces presented here (extending that of [4]) is new.

*Remark 5.1.16.* The restriction  $c_1 > 3/2$  is for technical reasons. The first time that it is actually needed is in our proof of the atomic decomposition theorem. It is possible to extend everything to  $c_1 > 1$  by a straightforward covering argument, but this would take extra work, and  $c_1 > 3/2$  is sufficient for our applications.

The following real interpolation theorem appears in [4, Theorem 2.9]. In Theorem 5.1.30 we will extend it to infinite exponents.

**Theorem 5.1.17** (Real interpolation for tent spaces with finite exponents). *Suppose that  $\mathbf{p}$  and  $\mathbf{q}$  are finite exponents with  $\theta(\mathbf{p}) \neq \theta(\mathbf{q})$ , and  $0 < \theta < 1$ . Then for all Whitney parameters  $c$  we have the identification*

$$(T^{\mathbf{p}}, T^{\mathbf{q}})_{\theta, p_\theta} = Z_c^{[\mathbf{p}, \mathbf{q}]_\theta}$$

*with equivalent quasinorms, where  $p_\theta = i([\mathbf{p}, \mathbf{q}]_\theta)$ .*

Consequently, when  $\mathbf{p}$  is finite, the  $Z$ -spaces  $Z_c^{\mathbf{p}}$  are complete and independent of  $c$  (up to equivalence of quasinorms). Hence we may simply write  $Z^{\mathbf{p}}$  in place of  $Z_c^{\mathbf{p}}$ . We will soon extend this to infinite exponents.

We will establish further properties of the  $Z$ -spaces ‘by hand’ rather than arguing by interpolation, because this yields stronger results. In particular, it yields absolute convergence of  $L^2$  duality pairings, while interpolation would only prove this on dense subspaces. This will be important in applications (Chapter 7). Our main tool is an equivalent dyadic characterisation of the  $Z^{\mathbf{p}}$ -quasinorm.<sup>4</sup> To establish this characterisation we will need some notation and a preliminary counting lemma.

For a standard (open) dyadic cube  $Q \in \mathcal{Q}(\mathbb{R}^n)$ ,<sup>5</sup> and for  $k \in \mathbb{Z}$ , define the *Whitney cube*

$$\overline{Q}^k := (2^k \ell(Q), 2^{k+1} \ell(Q)) \times Q,$$

and the *Whitney grid*

$$\mathcal{G}^k := \{\overline{Q}^k : Q \in \mathcal{Q}(\mathbb{R}^n)\}.$$

For each  $k \in \mathbb{Z}$ ,  $\mathcal{G}^k$  is a partition of  $\mathbb{R}_+^{1+n}$  up to a set of measure zero.

For each Whitney parameter  $c$ , each  $k \in \mathbb{Z}$ , and each Whitney cube  $\overline{Q}^k \in \mathcal{G}^k$ , we define

$$\mathcal{G}_c(\overline{Q}^k) := \{\overline{R}^k \in \mathcal{G}^k : \overline{R}^k \cap \Omega_c(t, x) \neq \emptyset \text{ for some } (t, x) \in \overline{Q}^k\}.$$

**Lemma 5.1.18.** *Let  $c$  be a Whitney parameter and  $k \in \mathbb{N}$ . Then for all  $\overline{Q}^k \in \mathcal{G}^k$  we have*

$$|\mathcal{G}_c(\overline{Q}^k)| \lesssim_{c,k,n} 1$$

(where  $|\cdot|$  denotes cardinality).

*Proof.* The condition  $\overline{R}^k \cap \Omega_c(t, x) \neq \emptyset$  may be rewritten as

$$\ell(R) \in [t/2^{k+1}c_1, 2^{-k}c_1t] \quad \text{and} \quad \text{dist}(R, x) < c_0t.$$

By rescaling and translating, the number of  $R \in \mathcal{Q}(\mathbb{R}^n)$  such that this condition is satisfied is equal to the number of  $R \in \mathcal{Q}(\mathbb{R}^n)$  such that

$$\ell(R) \in (1/2^{k+1}c_1, 2^{-k}c_1) \quad \text{and} \quad \text{dist}(R, 0) < c_0,$$

which is finite and depends only on  $c$ ,  $k$ , and  $n$ . □

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<sup>4</sup>This characterisation is stated and used by Barton and Mayboroda [21, Proof of Theorem 4.13], but without proof.

<sup>5</sup>Any system of dyadic cubes will work here.

**Proposition 5.1.19** (Dyadic characterisation). *Let  $\mathbf{p}$  be a finite exponent,  $c$  a Whitney parameter, and  $k \in \mathbb{Z}$ . Then we have*

$$\|f\|_{Z_c^{\mathbf{p}}} \simeq_{c,k,\mathbf{p}} \left\| \ell(Q)^{-r(\mathbf{p})} [|f|^2]_{\overline{Q}^k}^{1/2} \right\|_{\ell^p(\mathcal{G}^k, \ell(Q)^n)},$$

where  $[|f|^2]_{\overline{Q}^k}^{1/2} = \|f\|_{L^2(\overline{Q}^k, d\tau d\xi/\tau^{1+n})}$ .

*Proof.* Write  $\mathbf{p} = (p, s)$  and estimate

$$\begin{aligned} \|f\|_{Z_c^{\mathbf{p}}}^p &= \sum_{\overline{Q}^k \in \mathcal{G}^k} \iint_{\overline{Q}^k} \mathcal{W}_c(\kappa^{-s} f)(t, x)^p \frac{dt}{t} dx \\ &\simeq \sum_{\overline{Q}^k \in \mathcal{G}^k} (2^k \ell(Q))^{-ps} \iint_{\overline{Q}^k} \left\| f \mid L^2(\Omega_c(t, x), d\tau d\xi/\tau^{1+n}) \right\|^p \frac{dt}{t} dx \end{aligned} \quad (5.8)$$

$$\lesssim \sum_{\overline{Q}^k \in \mathcal{G}^k} (2^k \ell(Q))^{-ps} \iint_{\overline{Q}^k} \sum_{\overline{R}^k \in \mathcal{G}_c(\overline{Q}^k)} \left\| f \mid L^2(\overline{R}^k, d\tau d\xi/\tau^{1+n}) \right\|^p \frac{dt}{t} dx \quad (5.9)$$

$$\simeq_{k,p,s} \sum_{\substack{\overline{Q}^k \in \mathcal{G}^k \\ \overline{R}^k \in \mathcal{G}_c(\overline{Q}^k)}} \ell(R)^{n-ps} \left\| f \mid L^2(\overline{R}^k, d\tau d\xi/\tau^{1+n}) \right\|^p \quad (5.10)$$

$$\simeq \sum_{\overline{R}^k \in \mathcal{G}^k} \ell(R)^n \left( \ell(R)^{-s} \left\| f \mid L^2(\overline{R}^k, d\tau d\xi/\tau^{1+n}) \right\| \right)^p. \quad (5.11)$$

The equivalence (5.8) comes from the fact that  $\tau \simeq 2^k \ell(Q)$  when  $(\tau, \xi) \in \Omega_c(t, x)$  and  $(t, x) \in \overline{Q}^k$ . The upper bound (5.9) comes from covering  $\Omega_c(t, x)$  with the Whitney cubes  $\overline{R}^k \in \mathcal{G}_c(\overline{Q})$ , of which there are boundedly many by Lemma 5.1.18. The equivalence (5.10) comes from noting that  $\ell(R) \simeq \ell(Q)$  when  $\overline{R}^k \in \mathcal{G}_c(\overline{Q}^k)$ . Finally, (5.11) follows from the fact that every cube  $\overline{R}^k \in \mathcal{G}^k$  appears at least once, and at most a bounded number of times, in the multiset  $\{\overline{R}^k \in \mathcal{G}_c(\overline{Q}^k) : \overline{Q}^k \in \mathcal{G}\}$ .

To prove the converse statement, we need only prove the converse direction of (5.9). To do this we note that there exists a Whitney parameter  $\tilde{c}$  such that whenever  $\overline{R}^k \in \mathcal{G}_c(\overline{Q}^k)$  and  $(t, x) \in \overline{Q}^k$ , we have  $\overline{R}^k \subset \Omega_{\tilde{c}}(t, x)$ . Indeed, one can take  $\tilde{c}_0 = 2(c_0 + 2^{-k} \sqrt{n}(c_1 + 1))$  and  $\tilde{c}_1 = 4c_1$ . This, along with the independence of  $Z_c^{\mathbf{p}}$  on  $c$ , completes the proof.  $\square$

*Remark 5.1.20.* The same proof will work for infinite exponents once we show that the corresponding  $Z$ -space norms are independent of  $c$ .

The dyadic characterisation of the  $Z$ -space quasinorm can be used to prove a duality theorem when  $i(\mathbf{p}) > 1$ . As with the corresponding result for tent spaces, this is not just an abstract identification of dual spaces (which could be deduced by real interpolation), but also includes absolute convergence of the  $L^2$  duality pairing.

**Proposition 5.1.21** (Duality: reflexive range). *Suppose  $i(\mathbf{p}) \in (1, \infty)$ . Then for all  $f, g \in L^0(\mathbb{R}_+^{1+n})$  we have*

$$\iint_{\mathbb{R}_+^{1+n}} |(f(t, x), g(t, x))| dx \frac{dt}{t} \lesssim \|f\|_{Z^{\mathbf{p}}} \|g\|_{Z^{\mathbf{p}'}} , \quad (5.12)$$

and the  $L^2$  duality pairing identifies the Banach space dual of  $Z^{\mathbf{p}}$  with  $Z^{\mathbf{p}'}$ .

*Proof.* Let  $\overline{Q}_0 = (1, 2) \times (0, 1)^n$  equipped with the measure  $dx dt/t^{1+n}$ , and for each function  $f \in L^0(\mathbb{R}_+^{1+n})$  and each cube  $\overline{Q} \in \mathcal{G}$ , let  $f_{\overline{Q}}$  be the function on  $\overline{Q}_0$  which is the affine reparametrisation of  $\mathbf{1}_{\overline{Q}}f$ , so that  $[|f|^2]_{\overline{Q}}^{1/2} = \|f_{\overline{Q}}\|_{L^2(\overline{Q}_0)}$ . Then by Proposition 5.1.19, writing  $\mathbf{p} = (p, s)$ , we have

$$\begin{aligned} \|f\|_{Z^{\mathbf{p}}} &\simeq \left\| \ell(Q)^{-s} [|f|^2]_{\overline{Q}}^{1/2} \right\|_{\ell^p(\mathcal{G}, \ell(Q)^n)} \\ &\simeq \left\| \ell(Q)^{-s} f_{\overline{Q}} \right\|_{\ell^p(\mathcal{G}, \ell(Q)^n : L^2(\overline{Q}_0))} \\ &=: \|f_{\overline{Q}}\|_{\ell_s^p(\mathcal{G}, \ell(Q)^n : L^2(\overline{Q}_0))} . \end{aligned}$$

Evidently the map  $f \mapsto (f_{\overline{Q}})_{\overline{Q} \in \mathcal{G}}$  is an isomorphism between  $Z^{\mathbf{p}}$  and  $\ell_s^p(\mathcal{G}, \ell(Q)^n : L^2(\overline{Q}_0))$ .

Furthermore, for all  $f, g \in L^0(\mathbb{R}_+^{1+n})$  we have

$$\int_{\mathbb{R}_+^{1+n}} |(f(t, x), g(t, x))| dx \frac{dt}{t} \simeq \sum_{\overline{Q} \in \mathcal{G}} \ell(Q)^n \iint_{\overline{Q}_0} |(f_{\overline{Q}}(t, x), g_{\overline{Q}}(t, x))| \frac{dx dt}{t^{1+n}},$$

and so the mapping  $f \mapsto (f_{\overline{Q}})_{\overline{Q} \in \mathcal{G}}$  identifies the  $L^2(\mathbb{R}_+^{1+n})$  duality pairing with the  $\ell^2(\mathcal{G}, \ell(Q)^n : L^2(\overline{Q}_0))$  duality pairing (up to a constant).

Since we have

$$\sum_{\overline{Q} \in \mathcal{G}} \ell(Q)^n |(f_{\overline{Q}}, g_{\overline{Q}})_{L^2(\overline{Q}_0)}| \lesssim \|f_{\overline{Q}}\|_{\ell_s^p(\mathcal{G}, \ell(Q)^n : L^2(\overline{Q}_0))} \|g_{\overline{Q}}\|_{\ell_{-s}^{p'}(\mathcal{G}, \ell(Q)^n : L^2(\overline{Q}_0))}$$

and since the  $\ell^2(\mathcal{G}, \ell(Q)^n : L^2(Q_0))$  duality pairing identifies  $\ell_s^{p'}(\mathcal{G}, \ell(Q)^n : L^2(Q_0))$  as the dual of  $\ell_{-s}^p(\mathcal{G}, \ell(Q)^n : L^2(Q_0))$ , the corresponding results for  $Z^{\mathbf{p}}$  follow.  $\square$

The dyadic characterisation can also be used to prove an atomic decomposition theorem for  $Z$ -spaces.

**Definition 5.1.22.** Let  $\mathbf{p} = (p, s)$  be a finite exponent and  $c$  a Whitney parameter. We say that a function  $a \in L^0(\mathbb{R}_+^{1+n})$  is a  $Z_c^{\mathbf{p}}$ -atom associated with the point  $(t, x) \in \mathbb{R}_+^{1+n}$  if  $a$  is essentially supported in  $\Omega_c(t, x)$  and if

$$\left\| \kappa^{-s} a \right\|_{L^2(\Omega_c(t, x), dx dt/t)} \leq t^{n\delta_{p,2}}.$$

(recall that  $\delta_{p,2} = \frac{1}{2} - \frac{1}{p}$  is defined in Section 4.3).

**Lemma 5.1.23.** *Let  $\mathbf{p}$  be a finite exponent and suppose  $a$  is a  $Z_c^{\mathbf{p}}$ -atom associated with  $(t_0, x_0) \in \mathbb{R}_+^{1+n}$ . Then*

$$\|a\|_{Z^{\mathbf{p}}} \lesssim_{c, \mathbf{p}} 1.$$

*Proof.* A reasonably quick computation shows that

$$\{(t, x) \in \mathbb{R}_+^{1+n} : \Omega_c(t, x) \cap \Omega_c(t_0, x_0) \neq \emptyset\} \subset \Omega_{\tilde{c}}(t_0, x_0)$$

where  $\tilde{c}_0 = c_0(1 + c_1^2)$  and  $\tilde{c}_1 = c_1^2$ . Hence we can estimate, using the assumed support and size conditions for  $a$  and writing  $\mathbf{p} = (p, s)$ ,

$$\begin{aligned} & \|a\|_{Z_c^{\mathbf{p}}} \\ & \leq \left( \iint_{\Omega_{\tilde{c}}(t_0, x_0)} \left( \frac{1}{t^{1+n}} \iint_{\Omega_c(t_0, x_0)} \tau |\tau^{-s} a(\xi, \tau)|^2 d\xi \frac{d\tau}{\tau} \right)^{p/2} dx \frac{dt}{t} \right)^{1/p} \\ & \lesssim_c t_0^{n\delta_{p,2}} \left( \iint_{\Omega_{\tilde{c}}(t_0, x_0)} t_0^{-np/2} dx \frac{dt}{t} \right)^{1/p} \\ & \simeq_{c,p} t_0^{n\delta_{p,2} - \frac{n}{2} + \frac{n}{p}} \\ & = 1 \end{aligned}$$

as required.  $\square$

**Theorem 5.1.24** (Atomic decomposition of  $Z$ -spaces). *Suppose  $\mathbf{p} = (p, s)$  is a finite exponent with  $p \leq 1$  and  $c$  is a Whitney parameter. Then a function  $f \in L^0(\mathbb{R}_+^{1+n})$  is in  $Z^{\mathbf{p}}$  if and only if there exists a sequence  $(a_k)_k \in \mathbb{N}$  of  $Z_c^{\mathbf{p}}$ -atoms and a scalar sequence  $\lambda \in \ell^p(\mathbb{N})$  such that*

$$\sum_{k \in \mathbb{N}} \lambda_k a_k = f$$

*with convergence in  $Z^{\mathbf{p}}$ . Furthermore, we have*

$$\|f\|_{Z^{\mathbf{p}}} \simeq \inf \|\lambda\|_{\ell^p(\mathbb{N})},$$

*where the infimum is taken over all such decompositions.*

*Proof.* Given such a decomposition of  $f$ , we have

$$\|f\|_{Z^{\mathbf{p}}}^p = \left\| \sum_{k \in \mathbb{N}} \lambda_k a_k \right\|_{Z^{\mathbf{p}}}^p \lesssim \|\lambda\|_{\ell^p(\mathbb{N})}^p$$

by Lemma 5.1.23, and so  $\|f\|_{Z^{\mathbf{P}}} \lesssim \inf \|\lambda\|_{\ell^p(\mathbb{N})}$ . It remains to prove the reverse estimate. For each  $k \in \mathbb{Z}$  we can write

$$f = \sum_{\overline{Q}^k \in \mathcal{G}^k} f_{\overline{Q}^k} \quad (5.13)$$

where  $f_{\overline{Q}^k} = \mathbf{1}_{\overline{Q}^k} f$ ,<sup>6</sup> and by Proposition 5.1.19 this sum converges in  $Z^{\mathbf{P}}$ .<sup>7</sup> If  $k \geq \log_2(c_0^{-1}\sqrt{n}/3) + 1$  and if  $c_1 > 3/2$  (this is the first place where we actually use this assumption) then we have

$$\overline{Q}^k \subset \Omega_c(c_Q, t_Q)$$

for all  $Q \in \mathcal{Q}(\mathbb{R}^n)$ , where  $c_Q$  is the center of  $Q$  and  $t_Q$  is the midpoint of  $2^k \ell(Q)$  and  $2^{k+1} \ell(Q)$ . Therefore, under this condition on  $k$ , each  $f_{\overline{Q}^k}$  satisfies the support condition required of a  $Z_c^{\mathbf{P}}$ -atom. The norms  $\left\| \kappa^{-s} f_{\overline{Q}^k} \right\|_{L^2(\Omega_c(c_{\overline{Q}}, t_{\overline{Q}}), dx dt/t)}$  are all finite by Proposition 5.1.19, so we can define

$$\lambda_{\overline{Q}^k} := t_{\overline{Q}^k}^{-n\delta_{p,2}} \left\| \kappa^{-s} f_{\overline{Q}^k} \right\|_{L^2(\Omega_c(c_{\overline{Q}}, t_{\overline{Q}}), dx dt/t)}$$

and

$$a_{\overline{Q}^k} := \begin{cases} \lambda_{\overline{Q}^k}^{-1} f_{\overline{Q}^k} & (f_{\overline{Q}^k} \neq 0) \\ 0 & (f_{\overline{Q}^k} = 0). \end{cases}$$

Then each  $a_{\overline{Q}^k}$  is a  $Z_c^{\mathbf{P}}$ -atom and

$$f = \sum_{\overline{Q}^k \in \mathcal{G}^k} \lambda_{\overline{Q}^k} a_{\overline{Q}^k}$$

with convergence in  $Z^{\mathbf{P}}$ , and furthermore

$$\begin{aligned} \left\| (\lambda_{\overline{Q}^k}) \right\|_{\ell^p(\mathcal{G}^k)} &= \left\| t_{\overline{Q}^k}^{-n\delta_{p,2}} \left\| \kappa^{-s} f_{\overline{Q}^k} \right\|_{L^2(\Omega_c(c_{\overline{Q}^k}, t_{\overline{Q}^k}), dx dt/t)} \right\|_{\ell^p(\mathcal{G}^k)} \\ &\simeq \left\| (2^k \ell(Q))^{-n\delta_{p,2}-s} |Q|^{1/2} [|f|^2]_{\overline{Q}^k}^{1/2} \right\|_{\ell^p(\mathcal{G}^k)} \\ &\simeq \left\| \ell(Q)^{(n/p)-s} [|f|^2]_{\overline{Q}^k}^{1/2} \right\|_{\ell^p(\mathcal{G}^k)} \\ &\simeq \|f\|_{Z^{\mathbf{P}}} \end{aligned}$$

again using Proposition 5.1.19. □

<sup>6</sup>Note that this notation differs from that in the proof of Proposition 5.1.21.

<sup>7</sup>Convergence in  $Z^{\mathbf{P}}$  does not follow immediately: one must write the series (5.13) as a limit of partial sums and argue via dominated convergence.

*Remark 5.1.25.* In contrast to the setting of tent spaces, it is very easy to construct atomic decompositions of functions  $f \in Z^{\mathbf{p}}$ : as in the proof of the theorem, simply decompose  $f$  via the Whitney grid  $\mathcal{G}^k$  for sufficiently large  $k$ . This works for all finite  $\mathbf{p}$ , even if  $i(\mathbf{p}) > 1$ . Abstract decompositions will be used to prove  $Z^{\mathbf{p}}$ - $Z^{\mathbf{p}'}$  duality when  $i(\mathbf{p}) \leq 1$ .

**Lemma 5.1.26.** *For all Whitney parameters  $c$  and all  $f \in L^0(\mathbb{R}_+^{1+n})$ , the function  $\mathcal{W}_c f$  is lower semicontinuous.*

*Proof.* Fix  $M > 0$  and suppose that  $\mathcal{W}_c f(t, x) > M$ . Then there exists a small  $\varepsilon > 0$  such that  $\mathcal{W}_{(c_0-\varepsilon, c_1-\varepsilon)} f(t, x) > M$  also. A short computation shows that if  $\tilde{x} \in B(x, \varepsilon t/2)$  and if  $|\tilde{t} - t| < (c_1/(c_1 - \varepsilon) - 1)t$ , then  $\Omega_c(\tilde{t}, \tilde{x})$  contains  $\Omega_{(c_0-\varepsilon, c_1-\varepsilon)}(t, x)$ , so for all such  $(\tilde{t}, \tilde{x})$  we have

$$\mathcal{W}_c f(\tilde{t}, \tilde{x}) \geq \mathcal{W}_{(c_0-\varepsilon, c_1-\varepsilon)} f(t, x) > M.$$

Therefore the set  $\{(t, x) \in \mathbb{R}_+^{1+n} : \mathcal{W}_c f(t, x) > M\}$  is open.  $\square$

**Corollary 5.1.27.** *Let  $\mathbf{p}$  be an infinite exponent. Then*

$$\|f\|_{Z_c^{\mathbf{p}}} = \sup_{(t,x) \in \mathbb{R}_+^{1+n}} \mathcal{W}_c(\kappa^{-r(\mathbf{p})} f)(t, x),$$

*i.e. the essential supremum in the definition of the  $Z_c^{\mathbf{p}}$ -norm can be replaced with a supremum.*

*Proof.* Lower semicontinuity of the function  $\mathcal{W}_c(\kappa^{-r(\mathbf{p})} f)$  implies that if

$$\mathcal{W}_c(\kappa^{-r(\mathbf{p})} f)(t, x) > M$$

for some  $M < \infty$  at one point  $(t, x)$ , then it continues to hold in an open neighbourhood of  $(t, x)$ , and in particular on a set of positive measure.  $\square$

We can finally prove a duality theorem for  $Z^{\mathbf{p}}$  with  $i(\mathbf{p}) \leq 1$ . As with the other duality results so far, note that this includes absolute convergence of the  $L^2$  duality pairing.

**Theorem 5.1.28** (Duality: non-reflexive range). *Suppose  $i(\mathbf{p}) \leq 1$  and let  $c$  be a Whitney parameter. Then for all  $f, g \in L^0(\mathbb{R}_+^{1+n})$  we have*

$$\iint_{\mathbb{R}_+^{1+n}} |(f(t, x), g(t, x))| dx \frac{dt}{t} \lesssim \|f\|_{Z_c^{\mathbf{p}}} \|g\|_{Z_c^{\mathbf{p}'}} , \quad (5.14)$$

*and the  $L^2$  duality pairing identifies the Banach space dual of  $Z_c^{\mathbf{p}}$  with  $Z_c^{\mathbf{p}'}$ .*



*Proof.* Write  $\mathbf{p} = (p, s)$ , so that  $\mathbf{p}' = (\infty, -s, n\delta_{p,1})$ . First suppose  $a$  is a  $Z_c^{\mathbf{p}}$ -atom associated with a point  $(t_0, x_0) \in \mathbb{R}_+^{1+n}$ . Then we have

$$\begin{aligned} & \iint_{\mathbb{R}_+^{1+n}} |(a(t, x), g(t, x))| dx \frac{dt}{t} \\ & \leq \left\| \kappa^{-s} a \right\|_{L^2(\mathbb{R}_+^{1+n}, dx dt/t)} \left\| \kappa^s g \right\|_{L^2(\Omega_c(t_0, x_0), dx dt/t)} \\ & \lesssim t_0^{n\delta_{p,2} + n\delta_{1,p} + (n/2)} \left\| \kappa^{s-n\delta_{1,p}} g \right\|_{L^2(\Omega_c(t_0, x_0), dx dt/t^{1+n})} \\ & \leq \|g\|_{Z_c^{\mathbf{p}}} \end{aligned}$$

by Corollary 5.1.27. For general  $f \in Z^{\mathbf{p}}$ , write  $f$  as the sum of  $Z_c^{\mathbf{p}}$ -atoms as in Theorem 5.1.24, so that

$$\begin{aligned} \iint_{\mathbb{R}_+^{1+n}} |(f(t, x), g(t, x))| dx \frac{dt}{t} & \leq \sum_{k \in \mathbb{N}} |\lambda_k| \iint_{\mathbb{R}_+^{1+n}} |(a_k(t, x), g(t, x))| dx \frac{dt}{t} \\ & \lesssim \|g\|_{Z_c^{\mathbf{p}'}} \|\lambda\|_{\ell^p(\mathbb{N})} \end{aligned}$$

since  $p \leq 1$ . Taking the infimum over all atomic decompositions of  $f$  proves (5.14).

Now suppose that  $\phi \in (Z_c^{\mathbf{p}})'$ . By the same technique as in the proof of Proposition 5.1.21, we find that there exists a sequence  $(g_{\overline{Q}}) \in \ell_{-s}^\infty(\mathcal{G} : L^2(Q_0))$  corresponding to the induced action of  $\phi$  on  $\ell_s^p(\mathcal{G}, \ell(Q)^n : L^2(Q_0))$  (since  $\ell^p(\mathbb{N})' = \ell^\infty(\mathbb{N})$  for  $p \leq 1$ ). Hence there exists a function  $G_\phi \in L^0(\mathbb{R}_+^{1+n})$  corresponding to the action of  $\phi$  on  $Z_c^{\mathbf{p}}$ . We need to show that  $G_\phi$  is in  $Z_c^{\mathbf{p}'}$ .

Suppose  $(t, x) \in \mathbb{R}_+^{1+n}$ . Then we can estimate

$$\begin{aligned} \mathcal{W}_c(\kappa^{s-n\delta_{1,p}} G_\phi)(t, x) & \simeq t^{-n\delta_{1,p}} \|G_\phi\|_{L_{-s}^2(\Omega_c(t, x), d\xi d\tau/\tau^{1+n})} \\ & = t^{-n\delta_{1,p}} \sup_{\substack{F \in L_s^2(\Omega_c(t, x), d\xi d\tau/\tau^{1+n}) \\ \|F\| \leq 1}} |(F, G_\phi)| \\ & \lesssim t^{-n\delta_{1,p} - (n/2) - n\delta_{p,2}} \|\phi\|_{(Z_s^{\mathbf{p}})'} \sup_{\substack{F \in L_s^2(\Omega_c(t, x), d\xi d\tau/\tau) \\ \|F\| \leq t^{n\delta_{p,2}}}} \|F\|_{Z_s^{\mathbf{p}}} \\ & \leq \|\phi\|_{(Z^{\mathbf{p}})'} , \end{aligned}$$

using  $n\delta_{1,p} + (n/2) + n\delta_{p,2} = 0$ , the fact that the condition in the final supremum implies that  $F$  is a  $Z_c^{\mathbf{p}}$ -atom, and Lemma 5.1.23. Therefore we have

$$\|G_\phi\|_{Z_c^{\mathbf{p}'}} = \sup_{(t, x) \in \mathbb{R}_+^{1+n}} \mathcal{W}_c(\kappa^{s-n\delta_{1,p}} G_\phi)(t, x) \lesssim \|\phi\|_{(Z_c^{\mathbf{p}})'}$$

as desired.  $\square$

**Corollary 5.1.29.** *For all infinite exponents  $\mathbf{p}$  and all Whitney parameters  $c$ , the  $Z_s^{\mathbf{p}}$  norms are mutually equivalent.*

Hence for all exponents  $\mathbf{p}$  we write  $Z^{\mathbf{p}}$  in place of  $Z_c^{\mathbf{p}}$ .

Now that we have identified the duals of all  $Z^{\mathbf{p}}$  spaces for finite  $\mathbf{p}$ , we can give a full interpolation theorem.

**Theorem 5.1.30** (Real interpolation of tent spaces: full range). *Suppose that  $\mathbf{p}$  and  $\mathbf{q}$  are exponents with  $\theta(\mathbf{p}) \neq \theta(\mathbf{q})$ , and  $0 < \theta < 1$ . Then we have the identification*

$$(T^{\mathbf{p}}, T^{\mathbf{q}})_{\theta, p_{\theta}} = Z^{[\mathbf{p}, \mathbf{q}]_{\theta}}$$

*with equivalent quasinorms, where  $p_{\theta} = i([\mathbf{p}, \mathbf{q}]_{\theta})$ .*

*Proof.* For finite exponents this is precisely Theorem 5.1.17. If  $1 < i(\mathbf{p}), i(\mathbf{q}) \leq \infty$ , this follows by writing

$$(T^{\mathbf{p}}, T^{\mathbf{q}})_{\theta, p_{\theta}} = ((T^{\mathbf{p}'})', (T^{\mathbf{q}'})')_{\theta, p_{\theta}} = (T^{\mathbf{p}'}, T^{\mathbf{q}'})'_{\theta, p'_{\theta}}$$

via the duality theorem for real interpolation [22, Theorem 3.7.1], using that  $T^{\mathbf{p}'} \cap T^{\mathbf{q}'}$  is dense in both  $T^{\mathbf{p}'}$  and  $T^{\mathbf{q}'}$ , and then noting that

$$p'_{\theta} = i([\mathbf{p}', \mathbf{q}']_{\theta})' = i([\mathbf{p}, \mathbf{q}]_{\theta}).$$

The full result follows by Wolff reiteration [92, Theorem 1]. □

**Proposition 5.1.31** (Interpolation of  $Z$ -spaces). *Let  $\mathbf{p}$  and  $\mathbf{q}$  be exponents which are not both infinite, and let  $\theta \in (0, 1)$ . Then we have*

$$(Z^{\mathbf{p}}, Z^{\mathbf{q}})_{\theta, p_{\theta}} = Z^{[\mathbf{p}, \mathbf{q}]_{\theta}}$$

*with equivalent quasinorms, where  $p_{\theta} = i([\mathbf{p}, \mathbf{q}]_{\theta})$ . Furthermore if  $i(\mathbf{p}), i(\mathbf{q}) \geq 1$ , then we have*

$$[Z^{\mathbf{p}}, Z^{\mathbf{q}}]_{\theta} = Z^{[\mathbf{p}, \mathbf{q}]_{\theta}}.$$

*Proof.* The real interpolation result follows from Theorem 5.1.30 along with the reiteration theorem for real interpolation [22, Theorem 5.2.4]. The complex interpolation result is proven by Barton and Mayboroda via the dyadic characterisation of the norm [21, Theorem 4.13]. □

*Remark 5.1.32.* The  $Z$ -spaces can be seen as Wiener amalgam spaces  $W(L^2, L_w^p)$  associated to the semidirect product  $\mathbb{R}_+ \rtimes \mathbb{R}^n$  coming from the dilation action of

the multiplicative group  $\mathbb{R}_+$  on  $\mathbb{R}^n$ . Topologically  $\mathbb{R}_+ \ltimes \mathbb{R}^n = \mathbb{R}_+^{1+n}$ , and the group operation is given by  $(t, x) \cdot (s, y) := (t + s, x + ty)$ . Thus many of the properties above can be deduced from properties of abstract Wiener amalgam spaces. For a review of these spaces, see [45] and the references therein. However, if we were to use Wiener amalgam space arguments, we would not obtain any results for quasi-Banach  $Z$ -spaces (as the abstract theory of quasi-Banach Wiener amalgam spaces seems not to have been sufficiently developed), and we would not obtain absolute convergence of  $L^2$  duality pairings (only abstract duality pairings). Furthermore, these arguments would not show the connection with tent spaces.

#### 5.1.4 Unification: tent spaces, $Z$ -spaces, and slice spaces

Tent spaces and  $Z$ -spaces share the same fundamental properties. To make this totally explicit, we will write  $X$  as a placeholder for either  $T$  or  $Z$  when a statement holds for tent spaces and  $Z$ -spaces. When considering two different spaces, either of which can be a tent space or a  $Z$ -space independently, we will use subscripts  $X_0, X_1$ . For example, one can concisely write the conclusions of Theorem 5.1.30 and Proposition 5.1.31 as

$$(X^{\mathbf{p}}, X^{\mathbf{q}})_{\theta, p_\theta} = Z^{[\mathbf{p}, \mathbf{q}]_\theta},$$

and the tent space and  $Z$ -space duality results can be written extremely concisely as

$$(X^{\mathbf{p}})' = X^{\mathbf{p}'}$$

In this section we establish further properties of tent spaces and  $Z$ -spaces, including some interrelations between the two.

First, we simply point out that for all  $s \in \mathbb{R}$  we have

$$X_s^2 = L_s^2(\mathbb{R}_+^{1+n}),$$

where

$$\|f\|_{L_s^2} := \left\| \kappa^{-s} f \right\|_{L^2(\mathbb{R}_+^{1+n})}. \quad (5.15)$$

The following embedding theorem extends Theorem 5.1.11 not only to  $Z$ -spaces, but also to combinations of tent and  $Z$ -spaces.

**Theorem 5.1.33** (Mixed embeddings). *Let  $X_0, X_1 \in \{T, Z\}$  and let  $\mathbf{p} \hookrightarrow \mathbf{q}$  with  $\mathbf{p} \neq \mathbf{q}$ . Then we have the embedding*

$$(X_0)^{\mathbf{p}} \hookrightarrow (X_1)^{\mathbf{q}}.$$

*Proof.* When  $X_0 = X_1 = T$ , this is Theorem 5.1.11.

Let  $\mathbf{r} = [\mathbf{p}, \mathbf{q}]_2$ , so that  $\mathbf{p} \hookrightarrow \mathbf{r}$  and  $[\mathbf{p}, \mathbf{r}]_{1/2} = \mathbf{q}$  (by Lemmas 5.1.1 and 5.1.2). Then we have embeddings  $T^{\mathbf{p}} \hookrightarrow T^{\mathbf{p}}$  (trivially) and  $T^{\mathbf{p}} \hookrightarrow T^{\mathbf{r}}$  (Theorem 5.1.11). Therefore we have

$$T^{\mathbf{p}} \hookrightarrow (T^{\mathbf{p}}, T^{\mathbf{r}})_{1/2} = Z^{[\mathbf{p}, \mathbf{r}]_{1/2}} = Z^{\mathbf{q}}$$

by Theorem 5.1.30, using that  $\mathbf{p} \neq \mathbf{q}$  and  $\mathbf{p} \hookrightarrow \mathbf{q}$  imply  $\theta(\mathbf{p}) \neq \theta(\mathbf{q})$ . Similarly, putting  $\mathbf{s} = [\mathbf{p}, \mathbf{q}]_{-1}$ , we have  $T^{\mathbf{s}} \hookrightarrow T^{\mathbf{q}}$  and  $T^{\mathbf{q}} \hookrightarrow T^{\mathbf{s}}$ , so

$$Z^{\mathbf{p}} = (T^{\mathbf{s}}, T^{\mathbf{q}})_{1/2} \hookrightarrow T^{\mathbf{q}}.$$

Finally, putting  $\mathbf{t} = [\mathbf{p}, \mathbf{q}]_{1/2}$  and using the previous results, we have

$$Z^{\mathbf{p}} \hookrightarrow T^{\mathbf{t}} \hookrightarrow Z^{\mathbf{q}},$$

which completes the proof.  $\square$

We also have a convenient mixed embedding which only holds for infinite exponents.

**Lemma 5.1.34.** *Suppose that  $\mathbf{p}$  is infinite. Then  $T^{\mathbf{p}} \hookrightarrow Z^{\mathbf{p}}$ .*

*Proof.* Let  $f \in L^0(\mathbb{R}_+^{1+n})$  and write  $\mathbf{p} = (\infty, s; \alpha)$ . For  $\lambda > 0$  sufficiently large and for all  $(t, x) \in \mathbb{R}_+^{1+n}$ ,

$$\begin{aligned} & \left( \iint_{\Omega_{(1,2)}(t,x)} |\tau^{-(\alpha+s)} f(\tau, \xi)|^2 d\xi d\tau \right)^{1/2} \\ & \simeq \left( t^{-n-2\alpha} \iint_{\Omega_{(1,2)}(t,x)} |\tau^{-s} f(\tau, \xi)|^2 d\xi \frac{d\tau}{\tau} \right)^{1/2} \\ & \lesssim t^{-\alpha} \left( t^{-n} \iint_{T(B(x, \lambda t))} |\tau^{-s} f(\tau, \xi)|^2 d\xi \frac{d\tau}{\tau} \right)^{1/2}. \end{aligned}$$

Taking suprema over  $(t, x) \in \mathbb{R}_+^{1+n}$  yields

$$\|f\|_{Z^{\mathbf{p}}} \lesssim \|f\|_{T^{\mathbf{p}}}.$$

$\square$

**Proposition 5.1.35** (Density of intersections). *Let  $\mathbf{p}$  and  $\mathbf{q}$  be exponents, and let  $X_1, X_2 \in \{T, Z\}$ . If  $\mathbf{p}$  is finite then  $(X_1)^{\mathbf{p}} \cap (X_2)^{\mathbf{q}}$  is dense in  $(X_1)^{\mathbf{p}}$ . Otherwise,  $(X_1)^{\mathbf{p}} \cap (X_2)^{\mathbf{q}}$  is weak-star dense in  $(X_1)^{\mathbf{p}}$ .*

*Proof.* This follows immediately from the fact that  $L_c^2(\mathbb{R}_+^{1+n})$  is (weak-star) dense in  $T^{\mathbf{r}}$  for (infinite) exponents  $\mathbf{r}$ , and likewise in  $Z^{\mathbf{r}}$  (this can be proven directly, or by real interpolation, or by the embeddings of Theorem 5.1.33).  $\square$

For all  $r \in \mathbb{R}_+$ , define a ‘downward shift’ operator  $S_r$  on  $L^0(\mathbb{R}_+^{1+n})$  by

$$(S_r f)(t, y) := f(t + r, y)$$

for all  $f \in L^0(\mathbb{R}_+^{1+n})$ . These operators are well-behaved on certain tent spaces and  $Z$ -spaces, as shown in the following proposition.

**Proposition 5.1.36** (Uniform boundedness of downward shifts). *Let  $\mathbf{p}$  be an exponent.*

(i) *If  $i(\mathbf{p}) \leq 2$  and  $\theta(\mathbf{p}) < -1/2$ , then the operators  $(S_r)_{r \in \mathbb{R}_+}$  are uniformly bounded on  $X^{\mathbf{p}}$ .*

(ii) *If  $i(\mathbf{p}) \in (2, \infty]$  and  $r(\mathbf{p}) < -(n+1)/2$ , then the operators  $(S_r)_{r \in \mathbb{R}_+}$  are uniformly bounded on  $X^{\mathbf{p}}$ .*

*Remark 5.1.37.* Note that the assumptions for  $i(\mathbf{p}) \leq 2$  and  $i(\mathbf{p}) > 2$  are quite different: there is a sudden jump in dimensional dependence at  $i(\mathbf{p}) > 2$ . We do not currently have a good explanation for this behaviour, and there is no interpolation procedure to obtain stronger results when  $2 < i(\mathbf{p}) < \infty$ . Note that we can include endpoints when considering tent spaces (i.e. we can include  $\theta(\mathbf{p}) = -1/2$  or  $r(\mathbf{p}) = -(n+1)/2$  respectively). However, to realise the spaces  $Z^{\mathbf{p}}$  as interpolants of tent spaces, we need to interpolate between tent spaces  $T^{\mathbf{p}_0}$  and  $T^{\mathbf{p}_1}$  with  $\theta(\mathbf{p}_0) \neq \theta(\mathbf{p}_1)$ , and so the endpoint  $Z$ -space results cannot be proven by this argument.

*Proof.* It suffices to prove the tent space results; the  $Z$ -space results follow by real interpolation.

First we will prove boundedness on tent spaces for  $i(\mathbf{p}) \leq 1$  and for  $i(\mathbf{p}) = 2$ ; the rest of part (i) follows by complex interpolation. Suppose  $\mathbf{p} = (p, s)$  with  $p \leq 1$  and let  $a$  be a  $T^{\mathbf{p}}$ -atom associated with a ball  $B$  of radius  $r_B$ . Then  $S_r a$  is

supported on  $T(B)$ , and we have

$$\begin{aligned}
\|S_r a\|_{T_s^2} &\simeq \left( \int_0^{r_B-r} \int_{\mathbb{R}^n} t^{-2s-1} |a(t+r, x)|^2 dx dt \right)^{1/2} \\
&\leq \left( \int_0^{r_B-r} \int_{\mathbb{R}^n} (t+r)^{-2s-1} |a(t+r, x)|^2 dx dt \right)^{1/2} \\
&= \left( \int_r^{r_B} \int_{\mathbb{R}^n} |\tau^{-s} a(\tau, x)|^2 dx \frac{d\tau}{\tau} \right)^{1/2} \\
&\leq \|a\|_{T_s^2} \\
&\leq |B|^{\delta_{p,2}}
\end{aligned}$$

using that  $-2s-1 > 0$ . Therefore  $S_r a$  is, up to a uniform constant, a  $T^{\mathbf{p}}$ -atom associated with  $B$ . Hence if  $f = \sum_{k \in \mathbb{N}} \lambda_k a_k$  is an atomic decomposition of  $f$  in  $T^{\mathbf{p}}$ , then  $S_r f = \sum_{k \in \mathbb{N}} \lambda_k (S_r a_k)$  is an atomic decomposition of  $S_r f$  in  $T^{\mathbf{p}}$  up to a uniform constant. Therefore the operators  $(S_r)_{r \in \mathbb{R}_+}$  are uniformly bounded on  $T^{\mathbf{p}}$ . A similar argument (without needing atoms) works for  $\mathbf{p} = (2, s)$  provided  $s < -1/2$ .

Now let  $\mathbf{p} = (p, s)$  with  $p \in (2, \infty)$  and  $s < -(n+1)/2$ , and fix  $f \in L^0(\mathbb{R}_+^{1+n})$ . First we estimate  $S_r f$  in  $T^{\mathbf{p}}$ :

$$\begin{aligned}
\|S_r f\|_{T^{\mathbf{p}}} &= \left( \int_{\mathbb{R}^n} \left( \iint_{\Gamma(x)} t^{-2s-n-1} |f(t+r, y)|^2 dy dt \right)^{p/2} dx \right)^{1/p} \\
&\leq \left( \int_{\mathbb{R}^n} \left( \iint_{\Gamma(x)} (t+r)^{-2s-n-1} |f(t+r, y)|^2 dy dt \right)^{p/2} dx \right)^{1/p} \\
&= \left( \int_{\mathbb{R}^n} \left( \iint_{\Gamma(x)+r} \tau^{-2s-n-1} |f(\tau, y)|^2 dy d\tau \right)^{p/2} dx \right)^{1/p} \\
&\leq \|f\|_{T^{\mathbf{p}}}
\end{aligned}$$

using that  $-2s-n-1 > 0$  and  $\Gamma(x)+r \subset \Gamma(x)$ , where  $\Gamma(x)+r$  is the ‘vertically translated cone’

$$\Gamma(x)+r := \{(t, y) \in \mathbb{R}_+^{1+n} : (t-r, y) \in \Gamma(x)\}.$$

This proves part (ii) in the case where  $\mathbf{p}$  is finite.

Now suppose  $\mathbf{p} = (\infty, s; \alpha)$  with  $s+\alpha < -(n+1)/2$ , and let  $B = B(c, R) \subset \mathbb{R}^n$

be a ball. If  $r \leq R$ , then we can write

$$\begin{aligned}
& R^{-\alpha-n/2} \left( \iint_{T(B)} t^{-2s-1} |f(t+r, y)|^2 dy dt \right)^{1/2} \\
& \leq R^{-\alpha-n/2} \left( \iint_{T(B)} (t+r)^{-2s-1} |f(t+r, y)|^2 dy dt \right)^{1/2} \\
& \lesssim (R+r)^{-\alpha-n/2} \left( \iint_{T(B(c, R+r))} \tau^{-2s-1} |f(\tau, y)|^2 dy d\tau \right)^{1/2} \\
& \lesssim \|f\|_{T_s^\infty}
\end{aligned}$$

using that  $-2s-1 > 0$ . If  $r > R$  then instead we write

$$\begin{aligned}
& R^{-\alpha-n/2} \left( \iint_{T(B)} t^{-2s-1} |f(t+r, y)|^2 dy dt \right)^{1/2} \\
& = R^{-\alpha-n/2} \left( \iint_{T(B)+r} \tau^{-2s-1} \left( \frac{\tau-r}{\tau} \right)^{-2s-1} |f(\tau, y)|^2 dy d\tau \right)^{1/2} \\
& \leq R^{-\alpha-n/2} \left( \frac{R}{r} \right)^{-s-1/2} \left( \iint_{T(B)+r} \tau^{-2s-1} |f(\tau, y)|^2 dy d\tau \right)^{1/2} \\
& \leq \left( \frac{R+r}{R} \right)^{\alpha+n/2} \left( \frac{R}{r} \right)^{-s-1/2} \|f\|_{T_\alpha^\infty} \\
& \lesssim \|f\|_{T_\alpha^\infty}
\end{aligned}$$

using that  $s+\alpha \leq -(n+1)/2$  in the last line, where  $T(B)+r$  is defined analogously to  $\Gamma(x)+r$ . These estimates imply that  $\|S_r f\|_{T_\alpha^\infty} \lesssim \|f\|_{T_\alpha^\infty}$  as desired, completing the proof.  $\square$

Now we shall define the *slice spaces*. These were introduced in connection with tent spaces and boundary value problems by Auscher and Mourgoglou [15]. The name comes from the fact that functions in slice spaces are, roughly speaking, horizontal ‘slices’ of functions in tent or  $Z$ -spaces (this is made precise in Proposition 5.1.39).

**Definition 5.1.38.** Suppose  $\mathbf{p}$  is an exponent and  $t > 0$ . For  $f \in L^0(\mathbb{R}^n)$  we define

$$\|f\|_{E^{\mathbf{p}}(t)} := t^{-r(\mathbf{p})} \left\| x \mapsto \|f\|_{L^2(B(x,t), dy/t^n)} \right\|_{L^{i(\mathbf{p})}(\mathbb{R}^n)}.$$

These quasinorms define the *slice spaces*

$$E_{r(\mathbf{p})}^{i(\mathbf{p})}(t) = E^{\mathbf{p}}(t) = E^{\mathbf{p}}(t)(\mathbb{R}^n) := \{f \in L^0(\mathbb{R}^n) : \|f\|_{E^{\mathbf{p}}(t)} < \infty\}.$$

For  $t > 0$ ,  $h > 3/2$  (this technical restriction corresponds to that in the definition of Whitney parameter) and  $f \in L^0(\mathbb{R}^n)$ , define  $\iota_{t,h}(f) \in L^0(\mathbb{R}_+^{1+n})$  by setting

$$\iota_{t,h}(f)(s, x) := f(x) \mathbf{1}_{[t, ht]}(s)$$

for all  $(s, x) \in \mathbb{R}_+^{1+n}$ , and for  $g \in L^0(\mathbb{R}_+^{1+n})$  define  $\pi_t(g) \in L^0(\mathbb{R}^n)$  by

$$\pi_{t,h}(g)(x) := \int_t^{ht} g(s, x) \frac{ds}{s}.$$

for all  $x \in \mathbb{R}^n$ .

**Proposition 5.1.39.** *For all exponents  $\mathbf{p}$ , the operators*

$$E^{\mathbf{p}}(t) \xrightarrow{\iota_{t,h}} X^{\mathbf{p}} \xrightarrow{\pi_{t,h}} E^{\mathbf{p}}(t),$$

*are bounded uniformly in  $t$ . Furthermore, the compositions of these operators are identity maps.*

*Proof.* The tent space results with  $\theta(\mathbf{p}) = 0$  are already stated in [15, §3]; the extension to all tent spaces is simple. Likewise, the composition statement is clear. The proof for  $Z$ -spaces is a straightforward (one page) argument that we omit.  $\square$

Therefore we can view the spaces  $E^{\mathbf{p}}(t)$  as retracts of  $X^{\mathbf{p}}$ . Consequently, properties of tent spaces and  $Z$ -spaces descend to slice spaces.

**Proposition 5.1.40.** *If  $0 < t_0, t_1 < \infty$  and  $\mathbf{p}, \mathbf{q}$  are exponents with  $i(\mathbf{p}) = i(\mathbf{q})$ , then  $E^{\mathbf{p}}(t_0) = E^{\mathbf{q}}(t_1)$  with equivalent quasinorms.*

This follows from change of aperture for tent spaces (see [15, Lemma 3.5]). For  $p \in (0, \infty]$  we write  $E^p := E^{\mathbf{p}}(1)$  for any  $\mathbf{p}$  with  $i(\mathbf{p}) = p$ ; all  $E^{\mathbf{p}}(t)$  quasinorms are equivalent to the  $E^p$  quasinorm (but not uniformly in  $t$  or  $\mathbf{p}$ ).

We have a duality theorem for slice spaces, and of course one should notice once more that this includes absolute convergence of the  $L^2$  duality pairing (now on  $\mathbb{R}^n$  rather than  $\mathbb{R}_+^{1+n}$ ). This is proven in [15, Lemma 3.2].

**Proposition 5.1.41** (Duality). *Fix  $t > 0$  and let  $\mathbf{p}$  be a finite exponent.. Then we have*

$$\int_{\mathbb{R}^n} |(f(x), g(x))| dx \lesssim \|f\|_{E^{\mathbf{p}}} \|g\|_{E^{\mathbf{p}'}} , \quad (5.16)$$

*and the  $L^2(\mathbb{R}^n)$  duality pairing identifies the Banach space dual of  $E^{\mathbf{p}}$  with  $E^{\mathbf{p}'}$ .*



The tent space and  $Z$ -space embedding results also descend to slice spaces, though for slice spaces the ‘regularity’ parameters are not so important.

**Proposition 5.1.42** (Embeddings). *Suppose  $0 < p_0 \leq p_1 \leq \infty$ . Then  $E^{p_0} \hookrightarrow E^{p_1}$ .*

*Proof.* Fix  $\mathbf{p}_0$  and  $\mathbf{p}_1$  with  $i(\mathbf{p}_0) = p_0$ ,  $i(\mathbf{p}_1) = p_1$ , and  $\mathbf{p}_0 \hookrightarrow \mathbf{p}_1$ . Then we have bounded operators

$$E^{p_0} \xrightarrow{\iota_{1,2}} X^{\mathbf{p}_0} \hookrightarrow X^{\mathbf{p}_1} \xrightarrow{\pi_{1,2}} E^{p_1}$$

whose composition is the identity map, with the inclusion following from Theorem 5.1.33.  $\square$

Slice spaces contain the Schwartz functions, and are contained in the space of tempered distributions. This is contained in [15, Lemma 3.6].<sup>8</sup>

**Proposition 5.1.43.** *For all  $p \in (0, \infty]$  we have  $\mathcal{S} \subset E^p \subset \mathcal{S}'$ .*

We also have a straightforward integration by parts formula for functions in slice spaces. This is proven in [15, Lemma 3.8].

**Proposition 5.1.44** (Integration by parts in slice spaces). *Let  $\mathbf{p}$  be a finite exponent and suppose that  $\partial$  is a first-order differential operator with constant coefficients, and let  $\partial^*$  be the adjoint operator. If  $f, \partial f \in E^{\mathbf{p}}$  and  $g, \partial^* g \in E^{\mathbf{p}'}$ , then*

$$\int_{\mathbb{R}^n} (\partial f(x), g(x)) dx = \int_{\mathbb{R}^n} (f(x), \partial^* g(x)) dx.$$

Finally, we have an equivalent dyadic quasinorm for the slice spaces. This follows from the dyadic characterisation of  $Z$ -spaces (Proposition 5.1.19 and the remark following it) and Proposition 5.1.39.

**Proposition 5.1.45** (Dyadic characterisation). *For all  $p \in (0, \infty]$  we have*

$$\|f\|_{E^p} \simeq \left\| (\|f\|_{L^2(Q)})_{Q \in \mathcal{D}_1} \right\|_{\ell^p(\mathcal{D}_1)}$$

where  $\mathcal{D}_1$  is the grid of standard dyadic cubes in  $\mathbb{R}^n$  with sidelength 1.

*Remark 5.1.46.* The slice spaces  $E^p$  are equal to the Wiener amalgam spaces

$$W(L^2, L^p)(\mathbb{R}^n)$$

when  $p \geq 1$  (see [45] and the references therein). Therefore, as with  $Z$ -spaces, many properties of slice spaces can be deduced from properties of Wiener amalgam spaces. In order to emphasise the connection with tent spaces and  $Z$ -spaces, we have proven these results in this context.

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<sup>8</sup>Only the case  $p < \infty$  is included there, but everything (except the density statement) extends to  $p = \infty$ .

### 5.1.5 Homogeneous smoothness spaces

We will only give a quick definition of these, and state a few properties that we will need. These definitions are special cases of the Littlewood–Paley definitions of Triebel–Lizorkin and Besov spaces; we will not need these in full generality. For more information the reader can consult Grafakos [42, Chapter 6] or the many works of Triebel (for example [90, §5]).

Let  $\mathcal{Z}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$  be the set of Schwartz functions  $f$  such that  $D^\alpha f(0) = 0$  for every multi-index  $\alpha$ , and let  $\mathcal{Z}'(\mathbb{R}^n)$  be the topological dual of  $\mathcal{Z}(\mathbb{R}^n)$ . The space  $\mathcal{Z}'(\mathbb{R}^n)$  can be identified with the quotient space  $\mathcal{S}'(\mathbb{R}^n) \setminus \mathcal{P}(\mathbb{R}^n)$ , where  $\mathcal{P}(\mathbb{R}^n)$  is the space of polynomials on  $\mathbb{R}^n$ .

**Definition 5.1.47.** Let  $\Psi \in \mathcal{S}(\mathbb{R}^n)$  be a radial bump function with

$$\hat{\Psi} \geq 0, \quad \text{supp } \hat{\Psi} \subset A(0, 6/7, 2), \quad \text{and} \quad \hat{\Psi}|_{A(0,1,12/7)} = 1$$

(of course these precise parameters are not so important), and for  $j \in \mathbb{Z}$  let  $\Delta_j$  denote the associated Littlewood–Paley operators.

For  $f \in \mathcal{Z}'(\mathbb{R}^n)$ ,  $\alpha \in \mathbb{R}$ , and  $0 < p < \infty$  define

$$\|f\|_{\dot{H}_\alpha^p} := \left\| \left\| j \mapsto 2^{j\alpha} (\Delta_j f)(\cdot) \right\|_{\ell^2(\mathbb{Z})} \right\|_{L^p(\mathbb{R}^n)},$$

and for  $0 < p \leq \infty$  define

$$\|f\|_{\dot{B}_\alpha^{p,p}} := \left\| j \mapsto 2^{j\alpha} \|\Delta_j(f)\|_{L^p(\mathbb{R}^n)} \right\|_{\ell^p(\mathbb{Z})}.$$

The homogeneous *Hardy–Sobolev spaces*  $\dot{H}_\alpha^p = \dot{F}_\alpha^{p,2}$  and *Besov spaces*  $\dot{B}_\alpha^{p,p}$  are then the sets of those  $f \in \mathcal{Z}'(\mathbb{R}^n)$  for which the corresponding quasinorms are finite.

These quasinorms are independent of the choice of  $\Psi$  (up to equivalence), and  $\dot{H}_\alpha^p$  and  $\dot{B}_\alpha^{p,p}$  are Banach spaces (quasi-Banach when  $p < 1$ ).

For  $f \in \mathcal{Z}'(\mathbb{R}^n)$  and  $\alpha \in \mathbb{R}$ , the *Riesz potential*  $I_\alpha f \in \mathcal{Z}'(\mathbb{R}^n)$ , defined by

$$I_\alpha f(x) := (|\xi|^s \hat{f}(\xi))^\vee(x),$$

is well-defined. These operators can be used to characterise the Hardy–Sobolev spaces when  $p > 1$ .

**Theorem 5.1.48.** *Suppose  $1 < p < \infty$  and  $\alpha \in \mathbb{R}$ . Then  $f \in \mathcal{Z}'(\mathbb{R}^n)$  is  $\dot{H}_\alpha^p$  if and only if  $I_\alpha f \in L^p$ , and  $\|I_\alpha f\|_{L^p}$  is an equivalent norm on  $\dot{H}_\alpha^p$ . Furthermore, for all  $s \in \mathbb{R}$ ,  $I_\alpha$  is an isomorphism from  $\dot{H}_s^p$  to  $\dot{H}_{s+\alpha}^p$ .*

We will need characterisations of the Hardy–Sobolev and Besov spaces by integrals of differences. For all  $p \in [1, \infty]$ ,  $g \in L^0(\mathbb{R}^n)$ , and  $s \in \mathbb{R}$ , define

$$\mathcal{D}_s^p g(x) := \left( \int_{\mathbb{R}^n} \frac{|g(x+y) - g(x)|^p}{|y|^{n+ps}} dy \right)^{1/p} \quad (x \in \mathbb{R}^n).$$

**Lemma 5.1.49.** *Suppose  $\alpha \in (0, 1)$  and  $p \in (2n/(n + \alpha), \infty)$ . Then for all  $f \in L^2(\mathbb{R}^n)$  we have*

$$\|f\|_{\dot{H}_\alpha^p} \simeq \|\mathcal{D}_\alpha^2 f\|_{L^p}. \quad (5.17)$$

*Proof.* Whenever  $f = I_\alpha \varphi$  for some  $\varphi \in C_c^\infty$ , the estimate (5.17) follows from a lemma of Stein [85, Lemma 1] combined with the Riesz potential characterisation of  $\dot{H}_\alpha^p$  (Theorem 5.1.48). A density argument, using the fact that elements of  $\dot{H}_\alpha^p$  may be represented as  $L_{\text{loc}}^2$  functions when  $\alpha \in (0, 1)$ , completes the proof.  $\square$

The corresponding characterisation for Besov spaces can be found in [90, Theorem 5.2.3.2].

**Theorem 5.1.50.** *Suppose  $\alpha \in (0, 1)$  and  $p \in [1, \infty)$ . Then for all  $f \in L^2(\mathbb{R}^n)$  we have*

$$\|f\|_{\dot{B}_{\alpha}^{p,p}} \simeq \|\mathcal{D}_\alpha^p f\|_{L^p}.$$

For  $\alpha > 0$ , the Besov space  $\dot{B}_{\alpha}^{\infty,\infty}$  can be identified with the more familiar homogeneous *Hölder–Lipschitz space*  $\dot{\Lambda}_\alpha(\mathbb{R}^n)$ . We will only use this space when  $\alpha \in (0, 1)$ , and in this range  $\dot{\Lambda}_\alpha(\mathbb{R}^n)$  has a simple characterisation: it is the space of functions  $f$  on  $\mathbb{R}^n$  such that

$$\|f\|_{\dot{\Lambda}_\alpha} := \sup_{x,y \in \mathbb{R}^n} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty,$$

modulo constants. Such functions are automatically continuous.

We will also need to consider the Triebel–Lizorkin spaces  $\dot{F}_\alpha^{\infty,2}$  for  $\alpha \in \mathbb{R}$ , which are the subspaces of  $\mathcal{Z}'(\mathbb{R}^n)$  determined by the quasinorms

$$\|f\|_{\dot{F}_\alpha^{\infty,2}} := \inf \left\| \left\| j \mapsto 2^{j\alpha} |f_j(\cdot)| \right\|_{\ell^2(\mathbb{Z})} \right\|_{L^\infty(\mathbb{R}^n)},$$

with infima taken over all decompositions

$$f = \sum_{j \in \mathbb{Z}} \Delta_j f_j$$

with each  $f_j \in L^\infty(\mathbb{R}^n)$ , where  $\Delta_j$  are Littlewood–Paley operators as in Definition 5.1.47. When  $\alpha \geq 0$  these may be identified with the homogeneous BMO–Sobolev

spaces  $\dot{\mathbf{BMO}}_\alpha(\mathbb{R}^n)$ , which are defined as the images of  $\mathbf{BMO}(\mathbb{R}^n)$  under the Riesz potentials  $I_\alpha$  defined above, as subspaces of  $\mathcal{Z}'(\mathbb{R}^n)$ , with a corresponding norm. Of course  $\dot{\mathbf{BMO}}_0(\mathbb{R}^n) = \mathbf{BMO}(\mathbb{R}^n)$ . Information on these spaces can be found in [87] and [90, §5.1.4]. In particular, we have the following characterisation of  $\dot{\mathbf{BMO}}_\alpha(\mathbb{R}^n)$  for  $\alpha \in (0, 1)$  due to Strichartz [87, Theorem 3.3].

**Theorem 5.1.51.** *Suppose  $\alpha \in (0, 1)$ . Then for all  $f \in L^2(\mathbb{R}^n)$  we have*

$$\|f\|_{\dot{\mathbf{BMO}}_\alpha} \simeq \sup_Q \left( \frac{1}{|Q|} \int_Q \int_Q \frac{|f(x) - f(y)|^2}{|x - y|^{n+2\alpha}} dy dx \right)^{1/2},$$

where the supremum may be taken over all cubes or all balls.

We will introduce some unconventional but useful notation for these spaces. For a finite exponent  $\mathbf{p} = (p, s)$ , define

$$\mathbf{H}^{\mathbf{p}} := \dot{H}_s^p = \dot{F}_s^{p,2} \quad \text{and} \quad \mathbf{B}^{\mathbf{p}} := \dot{B}_s^{p,p}.$$

For  $\mathbf{p} = (\infty, s; 0)$ , define

$$\mathbf{H}^{\mathbf{p}} := \dot{F}_s^{\infty,2} \quad \text{and} \quad \mathbf{B}^{\mathbf{p}} := \dot{B}_s^{\infty,\infty}.$$

When  $s > 0$  we have  $\mathbf{H}^{\mathbf{p}} = \dot{\mathbf{BMO}}_s$  and  $\mathbf{B}^{\mathbf{p}} = \dot{\Lambda}_s$ . Finally, for  $\mathbf{p} = (\infty, s; \alpha)$  with  $\alpha > 0$ , define

$$\mathbf{H}^{\mathbf{p}} := \mathbf{B}^{\mathbf{p}} := \dot{B}_{s+\alpha}^{\infty,\infty}.$$

As a consequence of these definitions and the various duality identifications for classical smoothness spaces, for all finite exponents  $\mathbf{p}$  we have

$$(\mathbf{X}^{\mathbf{p}})' = \mathbf{X}^{\mathbf{p}'}$$

whenever  $\mathbf{X}$  denotes either  $\mathbf{H}$  or  $\mathbf{B}$ .

We also have the following interpolation theorem. This is a combination of standard results (see for example Mendez and Mitrea [73, Theorem 11], Triebel [88, Theorems 8.1.3 and 8.3.3a]), and Bergh and L fstr m [22, Theorem 6.4.5]).<sup>9</sup>

**Theorem 5.1.52.** *Let  $\mathbf{p}$  and  $\mathbf{q}$  be finite exponents, and suppose  $\theta \in (0, 1)$  and  $p_\theta := i([\mathbf{p}, \mathbf{q}]_\theta)$ . Then we have*

$$[\mathbf{H}^{\mathbf{p}}, \mathbf{H}^{\mathbf{q}}]_\theta = \mathbf{H}^{[\mathbf{p}, \mathbf{q}]_\theta}$$

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<sup>9</sup>The results cited in [88] and [22] are for inhomogeneous spaces. As always, essentially the same technique proves the result for homogeneous spaces. To obtain the stated results for Besov spaces with  $\theta(\mathbf{p}) = \theta(\mathbf{q})$ , write  $\dot{B}_\theta^{p,p} = \dot{F}_\theta^{p,p}$  and use the interpolation results for Triebel-Lizorkin spaces.

and (also allowing infinite exponents)

$$[\mathbf{B}^{\mathbf{p}}, \mathbf{B}^{\mathbf{q}}]_{\theta} = B^{[\mathbf{p}, \mathbf{q}]_{\theta}} \quad \text{and} \quad (\mathbf{B}^{\mathbf{p}}, \mathbf{B}^{\mathbf{q}})_{\theta, p_{\theta}} = B^{[\mathbf{p}, \mathbf{q}]_{\theta}}.$$

Furthermore if  $\theta(\mathbf{p}) \neq \theta(\mathbf{q})$ , then we have

$$(\mathbf{H}^{\mathbf{p}}, \mathbf{H}^{\mathbf{q}})_{\theta, p_{\theta}} = B^{[\mathbf{p}, \mathbf{q}]_{\theta}}.$$

## 5.2 Operator-theoretic preliminaries

### 5.2.1 Bisectorial operators and holomorphic functional calculus

The material of this section is not new, but we present it here to fix notation. Useful standard references are [70, 71, 2, 43], and a particularly nice recent exposition which focuses on bisectorial operators on Banach spaces is contained in the thesis of Egert [36, Chapter 3].

Let  $0 < \omega < \pi/2$ . The open bisector of angle  $\omega$  is the set

$$S_{\omega} := \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \omega \text{ or } |\arg(-z)| < \omega\} \subset \mathbb{C},$$

where the argument  $\arg(z)$  takes values in  $(-\pi, \pi]$ . The closed bisector of angle  $\omega$  is the topological closure  $\overline{S_{\omega}}$  of  $S_{\omega}$  in  $\mathbb{C}$ .

Throughout this section we will write  $L^2 = L^2(\mathbb{R}^n)$ .

**Definition 5.2.1.** Let  $0 \leq \omega < \pi/2$ . A closed linear operator  $A$  on  $L^2$  is called *bisectorial of angle  $\omega$*  if  $\sigma(A) \subset \overline{S_{\omega}}$ , and if for all  $\mu \in (\omega, \pi/2)$  and all  $z \in \mathbb{C} \setminus S_{\mu}$  we have the resolvent bound

$$\|(z - A)^{-1}\|_{\mathcal{L}(L^2)} \lesssim_{\mu} |z|^{-1}. \quad (5.18)$$

Note that closedness of  $A$  is included in this definition. This is not standard, but it is convenient. Generally the precise angle  $\omega$  is not important, in which case we will simply refer to  $A$  as *bisectorial*.

The following proposition is proven in [36, Proposition 3.2.2] (except for the adjoint statement, which is a simple computation).

**Proposition 5.2.2.** *Let  $A$  be a bisectorial operator on  $L^2$ . Then  $A$  is densely-defined, and we have a topological (not necessarily orthogonal) splitting*

$$L^2 = \mathcal{N}(A) \oplus \overline{\mathcal{R}(A)}. \quad (5.19)$$

Furthermore,  $A^*$  is also bisectorial.

The procedure of constructing an operator  $\varphi(A)$  from a given bisectorial operator  $A$  and holomorphic function  $\varphi$  on an appropriate bisector, known as *holomorphic functional calculus*, plays a central role in this work. In order to introduce holomorphic functional calculus properly, we must first define some classes of holomorphic functions.

For an angle  $\mu \in (0, \pi/2)$ , the set of holomorphic functions  $\varphi: S_\mu \rightarrow \mathbb{C}$  is denoted by  $H(S_\mu)$ . For  $\sigma, \tau \in \mathbb{R}$  and  $\varphi \in H(S_\mu)$  we define

$$\|\varphi\|_{\Psi_\sigma^\tau(S_\mu)} = \|z \mapsto \varphi(z)/m_\sigma^\tau(|z|)\|_{L^\infty(S_\mu)}$$

(the function  $m_\sigma^\tau$  is defined in Section 4.3) and

$$\Psi_\sigma^\tau(S_\mu) := \{\varphi \in H(S_\mu) : \|\varphi\|_{\Psi_\sigma^\tau(S_\mu)} < \infty\}.$$

Each  $\Psi_\sigma^\tau(S_\mu)$  is a Banach space when normed by  $\|\cdot\|_{\Psi_\sigma^\tau(S_\mu)}$ , and consists of those holomorphic functions on  $S_\mu$  which decay of order  $\sigma$  at 0 and of order  $\tau$  at  $\infty$ .<sup>10</sup> An important special case is  $\Psi_0^0(S_\mu) = H^\infty(S_\mu)$ , the set of bounded holomorphic functions on  $S_\mu$ . We will usually suppress reference to  $S_\mu$  in this notation, as the relevant bisector is generally clear from context.

The spaces  $\Psi_\sigma^\tau$  are decreasing in  $\sigma$  and  $\tau$ , in the sense that if  $\sigma < \sigma'$  and  $\tau < \tau'$ , then  $\Psi_{\sigma'}^{\tau'} \hookrightarrow \Psi_\sigma^\tau$ . For  $\sigma, \tau \in \mathbb{R}$  we define the set

$$\Psi_\sigma^{\tau+} := \bigcup_{\tau' > \tau} \Psi_\sigma^{\tau'},$$

and we define the sets  $\Psi_{\sigma+}^\tau$  and  $\Psi_{\sigma+}^{\tau+}$  analogously. The set  $\Psi_+^+ := \Psi_{0+}^{0+}$  is particularly important: it is the set of holomorphic functions (on the relevant bisector) with polynomial decay of some positive order at 0 and  $\infty$ . We also define

$$\Psi_\sigma^\infty := \bigcap_{\tau} \Psi_\sigma^\tau,$$

the set of functions with polynomial decay of arbitrarily large order at  $\infty$ . Similarly we can define  $\Psi_\infty^\tau$ ,  $\Psi_\infty^\infty$ ,  $\Psi_{\sigma+}^\infty$ , and so on.

There are a few holomorphic functions which we will use extensively. We define  $\chi^+, \chi^- \in H^\infty$  by

$$\chi^+(z) := \mathbf{1}_{z: \operatorname{Re}(z) > 0}(z) \quad \text{and} \quad \chi^-(z) := \mathbf{1}_{z: \operatorname{Re}(z) < 0}(z) \quad (z \in S_\mu); \quad (5.20)$$

these are the indicator functions of the two halves of the bisector  $S_\mu$ . We also define

$$[z] := \begin{cases} z & (\operatorname{Re}(z) > 0) \\ -z & (\operatorname{Re}(z) < 0) \end{cases} = (\chi^+(z) - \chi^-(z))z.$$

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<sup>10</sup>Decay of negative order is interpreted as growth.

This lets us define a bounded version of the exponential map,

$$\text{sgp} := [z \mapsto e^{-[z]}] \in \Psi_0^\infty, \quad (5.21)$$

which will be used to characterise solutions to Cauchy–Riemann systems in Chapter 7. For  $\lambda \in \mathbb{R} \setminus \{0\}$  we may also define the power function

$$[z \mapsto z^\lambda] \in \Psi_\lambda^{-\lambda}$$

via a branch cut on the half-line  $i(-\infty, 0] \subset \mathbb{C}$ .

We say that a function  $\varphi \in H(S_\mu)$  is *nondegenerate* if it does not vanish on any open subset of  $S_\mu$ . All the holomorphic functions defined above are nondegenerate except for  $\chi^+$  and  $\chi^-$ .

Let us introduce some useful operations on holomorphic functions. Let  $\varphi \in H(S_\mu)$ . There is a natural involution  $\varphi \mapsto \tilde{\varphi}$  on  $H(S_\mu)$  defined by

$$\tilde{\varphi}(z) := \overline{\varphi(\bar{z})} \quad (z \in S_\mu).$$

This involution is isometric on  $\Psi_\sigma^\tau$  for all  $\sigma, \tau \in \mathbb{R}$ . For  $t > 0$  we define the dilation  $\varphi_t \in H(S_\mu)$  by

$$\varphi_t(z) := \varphi(tz).$$

The following lemma is a simple consequence of the above definitions.

**Lemma 5.2.3.** *Let  $\sigma \in \mathbb{R}$ . Then for all  $t > 0$  we have*

$$\|\varphi_t\|_{\Psi_\sigma^{-\sigma}} = t^\sigma \|\varphi\|_{\Psi_\sigma^{-\sigma}}.$$

Fix an angle  $\omega \in [0, \pi/2)$  and let  $A$  be an  $\omega$ -bisectorial operator on  $L^2$ . If  $\mu \in (\omega, \pi/2)$  and  $\varphi \in \Psi_+^\tau(S_\mu)$ , then we can define an operator  $\varphi(A)$  on  $L^2$  by the Cauchy integral

$$\varphi(A)f := \frac{1}{2\pi i} \int_{\partial S_\nu} \varphi(z)(z - A)^{-1} f \, dz \quad (f \in L^2) \quad (5.22)$$

for any choice of  $\nu \in (\omega, \mu)$ , where  $\partial S_\nu$  is oriented counterclockwise. Then the integral (5.22) is well-defined and independent of the choice of  $\nu$ , and we have

$$\|\varphi(A)\|_{\mathcal{L}(L^2)} \lesssim_{A, \sigma, \tau, \mu} \|\varphi\|_{\Psi_\sigma^\tau(S_\mu)}.$$

The proof is straightforward; we mention only that independence of  $\nu$  follows from Cauchy’s integral theorem. The following homomorphism property also holds: when  $\varphi, \psi \in \Psi_+^\tau$ , we have  $(\varphi\psi)(A) = \varphi(A)\psi(A)$ . Straightforward

manipulations show that for all  $\psi \in \Psi_+^+$ , the adjoint operator  $\psi(A)^*$  is given by  $\tilde{\psi}(A^*)$ .

Often it is convenient to assume that the operator  $A$  is injective and has dense range. In the context in which we work this generally does not hold, but the splitting  $L^2 = \mathcal{N}(A) \oplus \overline{\mathcal{R}(A)}$  from Proposition 5.2.2 shows that the restriction  $A|_{\overline{\mathcal{R}(A)}}$ , acting on the Hilbert space  $\overline{\mathcal{R}(A)}$  (with inner product induced by that of  $L^2(\mathbb{R}^n)$ ), is injective and has dense range. One can also show that  $A|_{\overline{\mathcal{R}(A)}}$  is bisectorial.

The integral in (5.22) converges whenever  $\varphi \in \Psi_+^+$ , but if  $\varphi \in H^\infty$  is merely bounded, convergence is not guaranteed. We would like to be able to construct operators  $\varphi(A)$  when  $\varphi \in H^\infty$ . For certain operators this is possible.

**Definition 5.2.4.** Let  $A$  be a bisectorial operator on  $L^2$ . We say that  $A$  has *bounded  $H^\infty$  functional calculus on  $\overline{\mathcal{R}(A)}$*  if for all  $\varphi \in \Psi_+^+$  and all  $f \in \overline{\mathcal{R}(A)}$ , we have the estimate

$$\|\varphi(A)f\|_{L^2} \lesssim \|\varphi\|_\infty \|f\|_{L^2(\mathbb{R}^n)}.$$

The property of having bounded  $H^\infty$  functional calculus on  $\overline{\mathcal{R}(A)}$  is equivalent to certain quadratic estimates being satisfied; this is an important theorem due to McIntosh (see [70, §7 and §8] and the other references at the start of this section).

**Theorem 5.2.5** (McIntosh). *Let  $A$  be a bisectorial operator on  $L^2$ . Then  $A$  has bounded  $H^\infty$  functional calculus on  $\overline{\mathcal{R}(A)}$  if and only if the estimate*

$$\|f\|_{L^2} \simeq_\psi \left( \int_0^\infty \|\varphi_t(A)f\|_{L^2}^2 \frac{dt}{t} \right)^{1/2} \quad (5.23)$$

*holds for all  $f \in \overline{\mathcal{R}(A)}$  and some (equivalently, all) nondegenerate  $\varphi \in \Psi_+^+$ .*

Note that the quadratic estimate (5.23) need not hold for  $\varphi \in H^\infty$ .

If  $A$  has bounded  $H^\infty$  functional calculus on  $\overline{\mathcal{R}(A)}$ , then for all  $\varphi \in H^\infty$  we can define a bounded operator  $\varphi(A)$  on  $\overline{\mathcal{R}(A)}$  by

$$\varphi(A)f := \lim_\alpha \varphi_\alpha(A)f \quad (f \in \overline{\mathcal{R}(A)}), \quad (5.24)$$

where  $(\varphi_\alpha)$  is a net in  $\Psi_+^+$  which converges to  $\varphi$  in  $H^\infty$ . We then have

$$\|\varphi(A)\|_{\mathcal{L}(\overline{\mathcal{R}(A)})} \lesssim \|\varphi\|_{H^\infty}.$$

Furthermore, for  $\varphi, \psi \in H^\infty$ , we have the homomorphism property  $\varphi(A)\psi(A) = (\varphi\psi)(A)$ . For further details of this construction see [70, 71]. Thus we may define



bounded operators  $\chi^\pm(A)$  and  $e^{-t[A]} = \text{sgp}_t(A)$  (for all  $t > 0$ ) on  $\overline{\mathcal{R}(A)}$ , using the corresponding  $H^\infty$  functions defined in (5.20) and (5.21).

If  $\varphi \in \Psi_+^0$ , then we can extend  $\varphi(A)$  from  $\overline{\mathcal{R}(A)}$  to all of  $L^2$  by

$$\varphi(A)f := \varphi(A)\mathbb{P}_{\overline{\mathcal{R}(A)}}f,$$

where  $\mathbb{P}_{\overline{\mathcal{R}(A)}}$  is the projection onto  $\overline{\mathcal{R}(A)}$  associated with the decomposition  $L^2 = \mathcal{N}(A) \oplus \overline{\mathcal{R}(A)}$ . The operator  $\varphi(A)$  then maps  $L^2$  into  $\overline{\mathcal{R}(A)}$ . We have

$$\|\varphi(A)\|_{\mathcal{L}(L^2)} \lesssim \|\mathbb{P}_{\overline{\mathcal{R}(A)}}\|_{\mathcal{L}(L^2)} \|\varphi(A)\|_{\mathcal{L}(\overline{\mathcal{R}(A)})},$$

and furthermore the homomorphism property  $\psi(A)\varphi(A) = (\psi\varphi)(A)$  continues to hold for all  $\psi \in H^\infty$ .

We refer to the operators  $\chi^+(A)$  and  $\chi^-(A)$  as the *positive* and *negative spectral projections* associated with  $A$ . From the identities  $(\chi^\pm)^2 = \chi^\pm$ ,  $\chi^+\chi^- = 0$ , and  $\chi^+ + \chi^- = \mathbf{1}_{S_\mu}$  for the functions  $\chi^\pm$ , we deduce the identities

$$(\chi^\pm(A))^2 = \chi^\pm(A), \quad \chi^+(A)\chi^-(A) = \chi^-(A)\chi^+(A) = 0, \quad I_{\overline{\mathcal{R}(A)}} = \chi^+(A) + \chi^-(A)$$

for the spectral projections as operators on  $\overline{\mathcal{R}(A)}$ . Therefore  $\chi^+(A)$  and  $\chi^-(A)$  are complementary projections in  $\overline{\mathcal{R}(A)}$  onto the *positive* and *negative spectral subspaces*

$$\overline{\mathcal{R}(A)}^\pm := \chi^\pm(A)\overline{\mathcal{R}(A)},$$

and we have a topological direct sum decomposition

$$\overline{\mathcal{R}(A)} = \overline{\mathcal{R}(A)}^+ \oplus \overline{\mathcal{R}(A)}^-.$$

We define the *Cauchy operators*  $C_A^\pm : \overline{\mathcal{R}(A)} \rightarrow L^\infty(\mathbb{R}_\pm : \overline{\mathcal{R}(A)}^\pm)$  by

$$C_A^\pm f(t) := e^{-t[A]} \chi^\pm(A)f. \quad (5.25)$$

These are solution operators for the Cauchy problems associated with  $A$  on the upper half-space and lower half-space, in the following sense (see [8]).

**Proposition 5.2.6.** *Suppose that  $A$  has bounded  $H^\infty$  functional calculus on  $\overline{\mathcal{R}(A)}$ . If  $f \in \overline{\mathcal{R}(A)}^\pm$ , then  $F := C_A^\pm f$  solves the Cauchy problem*

$$\partial_t F(t) \pm AF(t) = 0, \quad F(0) = f$$

*in  $C^\infty(\mathbb{R}_\pm : \overline{\mathcal{R}(A)})$ .*

We end this section with a discussion of unbounded operators arising from holomorphic functional calculus, and some situations where their compositions are bounded.

Suppose that  $\varphi \in \Psi_\sigma^\tau$  with  $\min(\sigma, \tau) \leq 0$ , so that the integral (5.22) need not be absolutely convergent. We can define an unbounded operator  $\varphi(A)$  on  $\overline{\mathcal{R}(A)}$  as follows. Fix  $\delta > \max(-\sigma, -\tau) \geq 0$  and define  $\eta^\delta \in \Psi_\delta^\delta$  by

$$\eta^\delta(z) := \left( \frac{z}{(1+z)^2} \right)^\delta.$$

Then  $\eta^\delta \varphi \in \Psi_+^+$ , so that the operator  $(\eta^\delta \varphi)(A)$  is defined by (5.22). We also have  $\eta^\delta \in \Psi_+^+$ , so  $\eta^\delta(A)$  is also defined by (5.22), and since  $A$  is injective with dense range on  $\overline{\mathcal{R}(A)}$ , so is  $\eta^\delta(A)$ . Therefore the unbounded operator  $\eta^\delta(A)^{-1}$  is defined, with  $\mathcal{D}(\eta^\delta(A)^{-1}) := \mathcal{R}(\eta^\delta(A))$ . We then define the unbounded operator

$$\varphi(A) := \eta^\delta(A)^{-1}(\eta^\delta \varphi)(A) \quad (5.26)$$

with domain

$$\mathcal{D}(\varphi(A)) := \{f \in \overline{\mathcal{R}(A)} : (\eta^\delta \varphi)(A)f \in \mathcal{D}(\eta^\delta(A)^{-1})\}.$$

The operator  $\varphi(A)$  is closed, densely-defined, and independent of the choice of  $\delta$ . Of course, if  $\min(\sigma, \tau) > 0$ , then we can take  $\delta = 0$  in the definition (5.26) and recover the original definition of  $\varphi(A)$  by the Cauchy integral (5.22).

Now suppose  $\psi \in \Psi_{\sigma_1}^{\tau_1}$  and  $\varphi \in \Psi_{\sigma_2}^{\tau_2}$ . Then a quick computation shows that  $\varphi(A)\psi(A) \subseteq (\varphi\psi)(A)$ . Note that if  $\sigma_1 + \sigma_2 > 0$  and  $\tau_1 + \tau_2 > 0$ , then the operator  $(\varphi\psi)(A)$  is bounded and given by the Cauchy integral (5.22), while the operator  $\varphi(A)\psi(A)$  is not *a priori* given by such a representation. This observation will be convenient in what follows.

### 5.2.2 Off-diagonal estimates and the Standard Assumptions

For  $x \in \mathbb{R}$ , write  $\langle x \rangle := \max\{1, |x|\}$ . We continue to write  $L^2 = L^2(\mathbb{R}^n)$ .

**Definition 5.2.7.** Suppose  $\Omega \subset \mathbb{C} \setminus \{0\}$ , and let  $(S_z)_{z \in \Omega}$  be a family of operators in  $\mathcal{L}(L^2)$ . Let  $M \geq 0$ . We say that  $(S_z)$  satisfies *off-diagonal estimates of order  $M$*  if for all Borel subsets  $E, F \subset \mathbb{R}^n$ , all  $z \in \Omega$ , and all  $f \in L^2$ ,

$$\|\mathbf{1}_F S_z(\mathbf{1}_E f)\|_2 \lesssim \left\langle \frac{d(E, F)}{|z|} \right\rangle^{-M} \|\mathbf{1}_E f\|_2. \quad (5.27)$$

Many families of operators constructed from first-order differential operators (in particular, certain families of resolvents) satisfy off-diagonal estimates of some order. The following theorem shows that certain families constructed in terms of holomorphic functional calculus of a bisectorial operator  $A$  satisfy off-diagonal estimates, under the assumption that a certain resolvent family satisfies off-diagonal estimates. This is a slight extension of [84, Proposition 2.7.1]

**Theorem 5.2.8** (Off-diagonal estimates for families constructed by functional calculus). *Fix  $0 \leq \omega < \nu < \mu < \pi/2$ ,  $M \geq 0$ , and  $\sigma, \tau > 0$ . Let  $A$  be an  $\omega$ -bisectorial operator on  $L^2$  with bounded  $H^\infty$  functional calculus on  $\overline{\mathcal{R}(A)}$ , such that  $((I + \lambda A)^{-1})_{\lambda \in \mathbb{C} \setminus S_\nu}$  satisfies off-diagonal estimates of order  $M$ . Suppose that  $(\eta(t))_{t>0}$  is a continuous family of functions in  $H^\infty(S_\mu)$  which is uniformly bounded.<sup>11</sup> If  $\psi \in \Psi_\sigma^\tau(S_\mu)$ , then the family of operators  $(\eta(t)(A)\psi_t(A))_{t>0}$  satisfies off-diagonal estimates of order  $\min\{\sigma, M\}$ , with constants depending linearly on  $\|\psi\|_{\Psi_\sigma^\tau} \|\eta\|$ , where  $\|\eta\| := \sup_{t>0} \|\eta(t)\|_\infty$ , and also depending on  $A$ ,  $M$ ,  $\sigma$ , and  $\tau$ .*

*Proof.* Fix Borel sets  $E, F \subset \mathbb{R}^n$ . Because  $\psi_t(A)$  maps into  $\overline{\mathcal{R}(A)}$  for each  $t$ , we can apply  $\eta(t)(A)$  to  $\psi_t(A)(\mathbf{1}_E f)$  for each  $t > 0$ . We need to prove the estimate

$$\|\mathbf{1}_F \eta(t)(A) \psi_t(A)(\mathbf{1}_E f)\|_2 \lesssim_{A, M, \sigma, \tau} \|\eta\| \|\psi\|_{\Psi_\sigma^\tau(S_\mu)} \left\langle \frac{d(E, F)}{t} \right\rangle^{-\min\{\sigma, M\}} \|\mathbf{1}_E f\|_2$$

for all  $f \in L^2$ . Fix  $\nu' \in (\nu, \mu)$  throughout the proof.

If  $d(E, F) \leq t$ , then  $\langle d(E, F)/t \rangle \simeq 1$ , and so we have

$$\|\mathbf{1}_F \eta(t)(A) \psi_t(A)(\mathbf{1}_E f)\|_2 \leq \int_{\partial S_{\nu'}} |\eta(t)(z)| |\psi(tz)| \left\| \mathbf{1}_F(z - A)^{-1}(\mathbf{1}_E f) \right\|_2 |dz| \quad (5.28)$$

$$\begin{aligned} &\lesssim_A \|\eta\| \|\psi\|_{\Psi_\sigma^\tau} \|\mathbf{1}_E f\|_2 \int_{\partial S_{\nu'}} m_\sigma^\tau(|z|) \frac{|dz|}{|z|} \quad (5.29) \\ &\simeq_{\sigma, \tau, M} \|\eta\| \|\psi\|_{\Psi_\sigma^\tau} \left\langle \frac{d(E, F)}{t} \right\rangle^{-\min\{\sigma, M\}} \|\mathbf{1}_E f\|_2, \end{aligned}$$

where we used the resolvent bound coming from bisectoriality of  $A$  in (5.29).

Now suppose that  $d(E, F) > t$ . Then, rearranging (5.28) and using the as-

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<sup>11</sup>Continuity isn't really needed here - we only assume it to avoid measurability issues.

summed off-diagonal estimates for  $((I + \lambda A)^{-1})_{\lambda \in \mathbb{C} \setminus S_\nu}$ , we have

$$\begin{aligned}
& \| \mathbf{1}_F \eta(t)(A) \psi_t(A) (\mathbf{1}_E f) \|_2 \\
& \lesssim_A \| \eta \| \| \psi \|_{\Psi_\sigma^\tau} \| \mathbf{1}_E f \|_2 \int_{\partial S_{\nu'}} m_\sigma^\tau(|z|) \left\langle \frac{d(E, F)}{t/|z|} \right\rangle^{-M} \frac{|dz|}{|z|} \\
& \lesssim \| \eta \| \| \psi \|_{\Psi_\sigma^\tau} \| \mathbf{1}_E f \|_2 (\mathbf{I}_0 + \mathbf{I}_\infty), ,
\end{aligned} \tag{5.30}$$

where

$$\mathbf{I}_0 := \int_0^{td(E, F)^{-1}} m_\sigma^\tau(\lambda) \frac{d\lambda}{\lambda}$$

and

$$\mathbf{I}_\infty := \int_{td(E, F)^{-1}}^\infty m_\sigma^\tau(\lambda) \left( \frac{\lambda}{t} d(E, F) \right)^{-M} \frac{d\lambda}{\lambda}.$$

The integral  $\mathbf{I}_0$  is estimated by

$$\mathbf{I}_0 \leq \int_0^{td(E, F)^{-1}} \lambda^\sigma \frac{d\lambda}{\lambda} \simeq_\sigma \left( \frac{t}{d(E, F)} \right)^\sigma \lesssim_M \left\langle \frac{d(E, F)}{t} \right\rangle^{-\min(\sigma, M)}. \tag{5.31}$$

To estimate  $\mathbf{I}_\infty$ , we use that  $td(E, F)^{-1} \leq 1$  to write

$$\mathbf{I}_\infty \simeq_M \left\langle \frac{d(E, F)}{t} \right\rangle^{-M} \left( \int_{td(E, F)^{-1}}^1 \lambda^{\sigma-M} \frac{d\lambda}{\lambda} + C(\tau, M) \right)$$

where

$$C(\tau, M) = \int_1^\infty \lambda^{-\tau-M} \frac{d\lambda}{\lambda}.$$

If  $\sigma \leq M$ , then we have

$$\int_{td(E, F)^{-1}}^1 \lambda^{\sigma-M} \frac{d\lambda}{\lambda} \lesssim_{\sigma, M} \left( \frac{t}{d(E, F)} \right)^{\sigma-M},$$

and in this case

$$\begin{aligned}
\mathbf{I}_\infty & \lesssim_{\sigma, M} \left\langle \frac{d(E, F)}{t} \right\rangle^{-M} \left( \left\langle \frac{d(E, F)}{t} \right\rangle^{M-\sigma} + C(\tau, M) \right) \\
& \lesssim_{\tau, M} \left\langle \frac{d(E, F)}{t} \right\rangle^{-\min\{\sigma, M\}}.
\end{aligned} \tag{5.32}$$

Otherwise, we have

$$\int_{td(E, F)^{-1}}^1 \lambda^{\sigma-M} \frac{d\lambda}{\lambda} \lesssim_{\sigma, M} 1,$$

and this also yields (5.32). Putting the estimates (5.31) and (5.32) into (5.30) completes the proof.  $\square$

Off-diagonal estimates can also be used to deduce uniform boundedness and convergence results for families of operators on slice spaces. These propositions are proven in [15, §4].

**Proposition 5.2.9** (Uniform boundedness of families on slice spaces). *Let  $p \in (0, \infty]$ . If  $(T_s)_{s>0}$  is a family of operators on  $L^2$  satisfying off-diagonal estimates of order greater than  $n \min(|\delta_{p,2}|, 1/2)$ , then  $T_s$  extends to a bounded operator on  $E_0^p(t)$  uniformly in  $0 < s \leq t$ .*

**Proposition 5.2.10** (Strong convergence in slice spaces). *Let  $p \in (0, \infty)$ . Suppose  $(T_s)_{s>0}$  is a family of operators on  $L^2$  satisfying off-diagonal estimates of order greater than  $n \min(1/p, 1/2)$ , and such that  $\lim_{s \rightarrow 0} T_s = I$  strongly in  $L^2$ . Then  $\lim_{s \rightarrow 0} T_s = I$  strongly in  $E^p$ .*

Throughout the ‘abstract’ part of this work, the following assumptions will be sufficient. They can be a bit of a mouthful if stated in full, so we give them a name.

**Definition 5.2.11.** We say that an operator  $A$  satisfies the *Standard Assumptions* if

- $A$  is a  $\omega$ -bisectorial operator on  $L^2$  for some  $\omega \in [0, \pi/2)$ ,
- $A$  has bounded  $H^\infty$  functional calculus on  $\overline{\mathcal{R}(A)}$ , and
- for all  $\nu \in (\omega, \pi/2)$  the family  $((I + \lambda A)^{-1})_{\lambda \in \mathbb{C} \setminus S_\nu}$  satisfies off-diagonal estimates of arbitrarily large order.

The main examples we have in mind are perturbed Dirac operators.

**Theorem 5.2.12.** *Suppose  $D$  and  $B$  are as in Subsection 4.1.2 of the introduction. Then the perturbed Dirac operators  $DB$  and  $BD$  satisfy the standard assumptions (see Definition 5.2.11).*

See [17, Proposition 2.1] and [16, Lemma 2.3, Propositions 3.1 and 3.2]; the off-diagonal estimates stated there are in a different but equivalent form.

### 5.2.3 Integral operators on tent spaces

Let  $(S_{t,\tau})_{t,\tau>0}$  be a continuous two-parameter family of bounded operators on  $L^2 = L^2(\mathbb{R}^n)$ , and for all  $f \in L_c^2(\mathbb{R}_+ : L^2)$  define  $Sf \in L^0(\mathbb{R}_+ : L^2)$  by

$$Sf(t) := \int_0^\infty S_{t,\tau} f(\tau) \frac{d\tau}{\tau}. \quad (5.33)$$

Since  $f$  is compactly supported in  $\mathbb{R}_+$ , the Cauchy–Schwarz inequality shows that the integral (5.33) is absolutely convergent. We write  $S \sim (S_{t,\tau})_{t,\tau>0}$  to say that  $S$  is given by the kernel  $(S_{t,\tau})_{t,\tau>0}$ .

We would like know when  $S$  can be extended from  $L_c^2(\mathbb{R}_+ : L^2)$  to an operator between various tent spaces and  $Z$ -spaces. A first step is given by the following Schur-type lemma. Recall that  $L_s^2$  is defined in (5.15) and coincides with  $X_s^2$ .

**Lemma 5.2.13.** *Let  $s, \delta \in \mathbb{R}$ , and let  $S \sim (S_{t,\tau})_{t,\tau>0}$  on  $L^2$  as above. Suppose that there exists  $\gamma \in L^1(\mathbb{R}_+ : \mathbb{R})$  (where  $\mathbb{R}_+$  is equipped with the Haar measure  $dt/t$ ) such that for all  $t, \tau > 0$ ,*

$$\left\| \tau^{-\delta} S_{t,\tau} \right\|_{\mathcal{L}(L^2)} \leq \gamma(t/\tau) (t/\tau)^{s+\delta}.$$

Then for all  $f \in L_c^2(\mathbb{R}_+ : L^2)$ ,

$$\|Sf\|_{L_{s+\delta}^2} \lesssim \|\gamma\|_{L^1(\mathbb{R}_+)} \|f\|_{L_s^2}. \quad (5.34)$$

*Proof.* We argue by duality. For all  $g \in L_{-(s+\delta)}^2$ , we can estimate

$$\begin{aligned} & |\langle Sf, g \rangle| \\ & \leq \int_0^\infty \int_0^\infty \|S_{t,\tau} f(\tau)\|_2 \|g(t)\|_2 \frac{d\tau}{\tau} \frac{dt}{t} \\ & \leq \int_0^\infty \int_0^\infty \gamma(t/\tau) \left\| \tau^{-s} f(\tau) \right\|_2 \left\| t^{s+\delta} g(t) \right\|_2 \frac{d\tau}{\tau} \frac{dt}{t} \\ & \leq \left( \int_0^\infty \int_0^\infty \gamma(t/\tau) \frac{dt}{t} \left\| \tau^{-s} f(\tau) \right\|_2^2 \frac{d\tau}{\tau} \right)^{1/2} \\ & \quad \cdot \left( \int_0^\infty \int_0^\infty \gamma(t/\tau) \frac{d\tau}{\tau} \left\| t^{s+\delta} g(t) \right\|_2^2 \frac{dt}{t} \right)^{1/2} \\ & \leq \|\gamma\|_{L^1(\mathbb{R}_+)} \|f\|_{L_s^2} \|g\|_{L_{s+\delta}^2}, \end{aligned}$$

which implies (5.34).  $\square$

For certain kernels  $(S_{t,\tau})$ , assuming an  $L_s^2 \rightarrow L_{s+\delta}^2$  estimate (such as that which could be derived from the lemma above) and some off-diagonal estimates, we are able to deduce the boundedness of  $S$  from  $T^{\mathbf{p}}$  to  $T^{\mathbf{p}+\delta}$  for some exponents  $\mathbf{p}$  with  $i(\mathbf{p}) \in (0, 1]$ . This is a generalisation of an argument of Auscher, McIntosh, and Russ [13].

**Theorem 5.2.14** (Extrapolation of  $L^2$  boundedness to tent spaces). *Let  $\mathbf{p} = (p, s)$  be an exponent with  $p \leq 1$ , let  $\delta \in \mathbb{R}$ , and let  $(S_{t,\tau})_{t,\tau>0}$  be a continuous*

two-parameter family of bounded operators on  $L^2$  such that for all  $t_0, \tau_0 > 0$  the one-parameter families  $(t^{-\delta} S_{t, \tau_0})_{t \in (\tau_0, \infty)}$  and  $(\tau^{-\delta} S_{t_0, \tau})_{\tau \in (t_0, \infty)}$  both satisfy off-diagonal estimates of order  $M$ , with implicit constant  $K$  uniform in  $\tau_0$  and  $t_0$  respectively. Suppose  $a, b \in \mathbb{R}$ , and let  $\mathbf{S} \sim (m_a^b(t/\tau) S_{t, \tau})_{t, \tau > 0}$ .

If we have the norm estimate

$$\|\mathbf{S}f\|_{L_{s+\delta}^2} \lesssim \|f\|_{L_s^2} \quad (5.35)$$

for all  $f \in L_c^2(\mathbb{R}_+ : L^2)$ , with

$$-n\delta_{p,2} < b + s < M \quad \text{and} \quad a > s + \delta, \quad (5.36)$$

then

$$\|\mathbf{S}f\|_{T^{\mathbf{p}+\delta}} \lesssim \|f\|_{T^{\mathbf{p}}}$$

for all  $f \in L_c^2(\mathbb{R}_+ : L^2)$ , and the implicit constant is a linear combination of  $K$  and  $\|\mathbf{S}\| := \|\mathbf{S}\|_{L_s^2 \rightarrow L_{s+\delta}^2}$ .

**Proof. Step 1: an estimate for compactly supported atoms.** Suppose that  $f$  is a compactly-supported  $T^{\mathbf{p}}$ -atom associated with a ball  $B = B(c, r) \subset \mathbb{R}^n$ . Then  $f \in L_c^2$ , and so  $\mathbf{S}f$  is defined. We will show that  $\mathbf{S}f$  is in  $T^{\mathbf{p}+\delta}$  with quasinorm bounded independently of  $f$ . To do this we will exhibit an atomic decomposition of  $\mathbf{S}f$ , and we will estimate  $\|\mathbf{S}f\|_{T^{\mathbf{p}+\delta}}$  using the coefficients of this decomposition.

Let  $T_1 := T(4B)$  and  $T_k := T(2^{k+1}B) \setminus T(2^k B)$  for all integers  $k \geq 2$ . Then define  $F_k := \mathbf{1}_{T_k} \mathbf{S}f$  for all  $k \in \mathbb{N}$ , so that we have  $\mathbf{S}f = \sum_{k=1}^{\infty} F_k$  pointwise almost everywhere. For each  $k \in \mathbb{N}$  the function  $F_k$  is supported in a tent, so we can renormalise by writing  $F_k = \lambda_k f_k$  for some  $\lambda_k \in \mathbb{C}$  and some  $T^{\mathbf{p}+\delta}$ -atom  $f_k$ . We need only estimate the coefficients  $\lambda_k$ .

**Estimate for the local part.** For  $k = 1$ , since  $F_1$  is supported in  $T(4B)$ , we must estimate  $\|F_1\|_{L_{s+\delta}^2}$  in terms of  $|4B|^{\delta_{p,2}}$ . It follows from (5.35) and the fact that  $f$  is a  $T^{\mathbf{p}}$ -atom that

$$\|F_1\|_{L_{s+\delta}^2} \leq \|\mathbf{S}\| |B|^{\delta_{p,2}} \simeq_{n,p} \|\mathbf{S}\| |4B|^{\delta_{p,2}},$$

and so we can set  $\lambda_1 \simeq_{n,p} \|\mathbf{S}\|$ .

**Estimate for the global parts.** Suppose  $k \geq 2$ . Since  $F_k$  is supported in the tent  $T(2^{k+1}B)$ , we must estimate  $\|F_k\|_{L_{s+\delta}^2}$  in terms of  $|2^{k+1}B|^{\delta_{p,2}}$ . We use

Minkowski's integral inequality to estimate

$$\begin{aligned}
& \|F_k\|_{L^2_{s+\delta}} \\
&= \left( \int_0^{2^{k+1}r} \left\| t^{-(s+\delta)} \mathbf{1}_{B(c, 2^{k+1}r-t)} \int_0^r m_a^b(t/\tau) S_{t,\tau} f(\tau) \frac{d\tau}{\tau} \right\|_{L^2}^2 \right)^{1/2} \\
&\leq \left( \int_0^{2^{k+1}r} \left( \int_0^r t^{-(s+\delta)} m_a^b(t/\tau) \left\| \mathbf{1}_{A(c, 2^k r-t, 2^{k+1}r-t)} S_{t,\tau} f(\tau) \right\|_{L^2} \frac{d\tau}{\tau} \right)^2 \frac{dt}{t} \right)^{1/2} \\
&\leq \int_0^r \left( \int_0^{2^{k+1}r} t^{-2(s+\delta)} m_a^b(t/\tau)^2 \left\| \mathbf{1}_{A(c, 2^k r-t, 2^{k+1}r-t)} S_{t,\tau} f(\tau) \right\|_{L^2}^2 \frac{dt}{t} \right)^{1/2} \frac{d\tau}{\tau}.
\end{aligned}$$

Note that  $f(\tau)$  is supported in  $B(c, r - \tau)$ . We have

$$\begin{aligned}
d(\text{supp } f(\tau), A(c, 2^k r - t, 2^{k+1}r - t)) &\geq d(B(c, r), \mathbb{R}^n \setminus B(c, 2^k r - t)) \\
&= ((2^k - 1)r - t)_+,
\end{aligned}$$

so we split the region of integration  $(0, 2^{k+1}r) \times (0, r)$  into three subregions,

$$\begin{aligned}
R_1 &:= \{(t, \tau) : t < \tau < r\} \\
R_2 &:= \left\{ (t, \tau) : \tau < t < \frac{2^k - 1}{2} r \right\} \\
R_3 &:= \left\{ (t, \tau) : t > \frac{2^k - 1}{2} r \right\},
\end{aligned}$$

and denote the corresponding integrals by  $\mathbf{I}_1$ ,  $\mathbf{I}_2$ , and  $\mathbf{I}_3$ .<sup>12</sup>

On  $R_3$ , where  $t > \tau$  and where there is no spatial separation, we have

$$\begin{aligned}
\mathbf{I}_3 &\lesssim K \int_0^r \left( \int_{\frac{2^k-1}{2}r}^{2^{k+1}r} t^{-2(s+\delta)} (t/\tau)^{-2b} t^{2\delta} \|f(\tau)\|_{L^2}^2 \frac{dt}{t} \right)^{1/2} \frac{d\tau}{\tau} \\
&\lesssim_{b,s} K \int_0^r \left\| \tau^{-s} f(\tau) \right\|_{L^2} \tau^{b+s} (2^k r)^{-(b+s)} \frac{d\tau}{\tau} \\
&\leq K 2^{-k(b+s)} \left( \int_0^r \left\| \tau^{-s} f(\tau) \right\|_{L^2}^2 \frac{d\tau}{\tau} \right)^{1/2} \left( \int_0^r \left( \frac{\tau}{r} \right)^{2(b+s)} \frac{d\tau}{\tau} \right)^{1/2} \\
&\simeq_{b,s} K 2^{-k(b+s)} \|f\|_{L^2_s} \\
&\leq K 2^{-k(b+s)} |B|^{\delta_{p,2}} \\
&= K 2^{-k(b+s)} |2^{k+1} B|^{\delta_{p,2}} \left( \frac{|2^{k+1} B|}{|B|} \right)^{-\delta_{p,2}} \\
&\simeq K 2^{-k(b+s+n\delta_{p,2})} |2^{k+1} B|^{\delta_{p,2}},
\end{aligned} \tag{5.37}$$

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<sup>12</sup>The reason for using the factor  $(2^k - 1)/2$  rather than  $2^k - 1$  will be apparent when estimating  $\mathbf{I}_2$ .



where we used  $b + s > -n\delta_{p,2} > 0$  in (5.37).

On  $R_1$ , where  $t < \tau$  and where the off-diagonal estimates for  $(\tau^{-\delta} S_{t,\tau})_{\tau \in (t,\infty)}$  involve spatial separation,

$$\begin{aligned} \mathbf{I}_1 &\lesssim K \int_0^r \left( \int_0^\tau t^{-2(s+\delta)} (t/\tau)^{2a} \left( \left( \frac{(2^k - 1)r - t}{\tau} \right)^{-M} \|f(\tau)\|_{L^2} \right)^2 \frac{dt}{t} \right)^{1/2} \tau^\delta \frac{d\tau}{\tau} \\ &\simeq_{a,s,\delta} K 2^{-kM} \int_0^r \left( \frac{\tau}{r} \right)^M \left\| \tau^{-s} f(\tau) \right\|_{L^2} \frac{d\tau}{\tau} \end{aligned} \quad (5.38)$$

$$\begin{aligned} &\leq K 2^{-kM} \|f\|_{L^2_s} \left( \int_0^r \left( \frac{\tau}{r} \right)^{2M} \frac{d\tau}{\tau} \right)^{1/2} \\ &\lesssim_M K 2^{-k(M+n\delta_{p,2})} |2^{k+1} B|^{\delta_{p,2}} \end{aligned} \quad (5.39)$$

using that  $a > s + \delta$  in (5.38), and deducing (5.39) from the same argument used for  $\mathbf{I}_3$ .

On  $R_2$ , we have  $t > \tau$  and the off-diagonal estimates for  $(t^{-\delta} S_{t,\tau})_{t \in (\tau,\infty)}$  again involve spatial separation, and the restrictions on  $t$  imply  $(2^k - 1)r - t > \frac{2^k - 1}{2}r \gtrsim 2^k r$ . Therefore

$$\begin{aligned} \mathbf{I}_2 &\lesssim K \int_0^r \left( \int_\tau^{\frac{2^k - 1}{2}r} t^{-2s} (t/\tau)^{-2b} \left( \frac{(2^k - 1)r - t}{t} \right)^{-2M} \|f(\tau)\|_{L^2}^2 \frac{dt}{t} \right)^{1/2} \frac{d\tau}{\tau} \\ &\lesssim K \int_0^r \left\| \tau^{-s} f(\tau) \right\|_{L^2} \tau^{b+s} \left( \int_\tau^{\frac{2^k - 1}{2}r} t^{-2(b+s)} \left( \frac{2^k r}{t} \right)^{-2M} \frac{dt}{t} \right)^{1/2} \frac{d\tau}{\tau} \\ &\lesssim_{b,s,M} K 2^{-kM} \int_0^r \left\| \tau^{-s} f(\tau) \right\|_{L^2} \tau^{b+s} r^{-M} (2^k r)^{-(b+s-M)} \frac{d\tau}{\tau} \end{aligned} \quad (5.40)$$

$$\lesssim_{b,s} K 2^{-k(b+s+n\delta_{p,2})} |2^{k+1} B|^{\delta_{p,2}} \quad (5.41)$$

using that  $M > s_0 + b$  in (5.40) and arguing as before to conclude (5.41).

Summing up, we have

$$\begin{aligned} \|F_k\|_{L^2_{s+\delta}} &\leq \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 \\ &\lesssim_{a,b,M,p,s,\delta} K \left( 2^{-k(M+n\delta_{p,2})} + 2^{-k(b+s+n\delta_{p,2})} \right) |2^{k+1} B|^{\delta_{p,2}}, \end{aligned}$$

and so for  $k \geq 2$  we can set

$$\lambda_k \simeq K \left( 2^{-k(M+n\delta_{p,2})} + 2^{-k(b+s+n\delta_{p,2})} \right),$$

which implies that

$$\begin{aligned} \|(\lambda_k)\|_{\ell^p(\mathbb{N})}^p &\simeq \|\mathbf{S}\|^p + K^p \sum_{k=2}^{\infty} \left( 2^{-k(M+n\delta_{p,2})} + 2^{-k(b+s+n\delta_{p,2})} \right)^p \\ &\leq \|\mathbf{S}\|^p + K^p \sum_{k=2}^{\infty} 2^{-kp(M+n\delta_{p,2})} + 2^{-kp(b+s+n\delta_{p,2})}, \end{aligned}$$

which is finite because of the assumption (5.36). The implicit constants do not depend on the atom  $f$ . Therefore  $\mathbf{S}f$  is in  $T_{s_1}^p$ , with quasinorm bounded independently of  $f$  and controlled by a linear combination of  $\|\mathbf{S}\|$  and  $K$ .

**Step 2: from compactly supported atoms to  $T_s^p \cap L_c^2(\mathbb{R}_+ : L^2)$ .** This final part of the argument exactly follows [13, Proof of Theorem 4.9, Step 3]. One must show that every function in  $T_s^p \cap L_c^2(\mathbb{R}_+ : L^2)$  may be decomposed into a sum of compactly supported atoms, and that such decompositions converge in both  $T_s^p$  (which is automatic) and in  $L_s^2$ . We omit further details.  $\square$

#### 5.2.4 Extension and contraction operators

Throughout this section we assume that  $A$  is an operator which satisfies the standard assumptions (see Definition 5.2.11).

**Definition 5.2.15.** For all  $\psi \in H^\infty$  define the *extension operator*

$$\mathbb{Q}_{\psi,A} : \overline{\mathcal{R}(A)} \rightarrow L^\infty(\mathbb{R}_+ : L^2)$$

by

$$(\mathbb{Q}_{\psi,A}f)(t) := \psi_t(A)f \quad (f \in \overline{\mathcal{R}(A)}, t \in \mathbb{R}_+).$$

If in addition  $\psi \in \Psi_+^+$ , then  $\mathbb{Q}_{\psi,A}$  is defined on all of  $L^2$ , and by Theorem 5.2.5 we have boundedness  $\mathbb{Q}_{\psi,A} : L^2 \rightarrow L^2(\mathbb{R}_+^{1+n})$ .

**Definition 5.2.16.** For all  $\varphi \in \Psi_+^+$  define the *contraction operator*

$$\mathbb{S}_{\varphi,A} : L^2(\mathbb{R}_+^{1+n}) \rightarrow \overline{\mathcal{R}(A)}$$

by

$$\mathbb{S}_{\varphi,A} := (\mathbb{Q}_{\tilde{\varphi},A^*})^*.$$

Note that  $\mathbb{Q}_{\psi,A} = (\mathbb{S}_{\tilde{\psi},A^*})^*$  when  $\psi \in \Psi_+^+$ .

A quick computation yields the following representation of  $\mathbb{S}_{\varphi,A}$ . The integral in (5.42) converges absolutely since  $f \in L^1(\mathbb{R}_+ : L^2(\mathbb{R}^n))$ .

**Proposition 5.2.17.** *Suppose  $\varphi \in \Psi_+^+$ . Then for all  $f \in L_c^2(\mathbb{R}_+ : L^2)$  we have*

$$\mathbb{S}_{\varphi,A}f := \int_0^\infty \varphi_t(A)f(t) \frac{dt}{t}. \quad (5.42)$$

Fix  $\delta \in \mathbb{R}$ , and suppose  $\eta \in \Psi_{-\delta}^\delta$ ,  $\psi \in H^\infty$ , and  $\varphi \in \Psi_\delta^{-\delta} \cap \Psi_+^+$ . Then for all  $f \in L_c^2(\mathbb{R}_+ : L^2)$  we have  $\mathbb{S}_{\varphi,A}f \in \mathcal{D}(\eta(A))$  and the integral representation

$$(\mathbb{Q}_{\psi,A}\eta(A)\mathbb{S}_{\varphi,A}f)(t) = \int_0^\infty (\psi_t(A)\eta(A)\varphi_\tau(A))f(\tau) \frac{d\tau}{\tau}. \quad (5.43)$$

Therefore we can write

$$\mathbb{Q}_{\psi,A}\eta(A)\mathbb{S}_{\varphi,A} \sim ((\psi_t\eta\varphi_\tau)(A))_{t,\tau>0}.$$

Our goal now is to check when the results of Section 5.2.3 apply to this operator. In fact, we will be able to draw some conclusions even when  $\eta$  is not bounded, as long as  $\psi_t\eta\varphi_\tau \in \Psi_+^+$ . This will ultimately lead to Theorem 6.1.11.

**Lemma 5.2.18.** *Suppose  $\sigma + \tau \geq 0$  and  $\delta \in \mathbb{R}$ . Let  $\psi \in \Psi_\sigma^\tau$ ,  $\varphi \in \Psi_{\tau+\delta}^{\sigma-\delta}$ , and  $\eta \in \Psi_{-\delta}^\delta$ , and define the operator*

$$\tilde{S}_{t,r} := m_\sigma^{\tau+\delta}(t/r)^{-1}(\psi_t\eta\varphi_r)(A). \quad (5.44)$$

*Then for all  $t_0, r_0 > 0$  the operator families  $(t^{-\delta}\tilde{S}_{t,r_0})_{t \in (r_0, \infty)}$  and  $(r^{-\delta}\tilde{S}_{t_0,r})_{r \in (t_0, \infty)}$  satisfy off-diagonal estimates of order  $\sigma + \tau$ , uniformly in  $r_0$  and  $t_0$  respectively. The implicit constants in these off-diagonal estimates depend linearly on  $\|\eta\|_{\Psi_{-\delta}^\delta}$ .*

This is a variation of [13, Lemma 3.7].

*Proof.* If  $t_0 \leq r$  we can write

$$\begin{aligned} r^{-\delta}\tilde{S}_{t_0,r} &= r^{-\delta}(t_0/r)^{-\sigma}[\psi_{t_0}(z)\eta(z)\varphi_r(z)](A) \\ &= [(t_0z)^{-\sigma}\psi_{t_0}(z)\eta^\delta(z)(rz)^{\sigma-\delta}\varphi_r(z)](A) \end{aligned}$$

where  $\eta^\delta \in H^\infty$  is defined by  $\eta^\delta(z) := z^\delta\eta(z)$ . Note that  $\|\eta^\delta\|_\infty = \|\eta\|_{\Psi_{-\delta}^\delta}$ . Since  $\psi \in \Psi_\sigma^\tau$  and  $\sigma + \tau \geq 0$ , the function

$$\gamma(t_0) : z \mapsto (t_0z)^{-\sigma}\psi_{t_0}(z)\eta^\delta(z)$$

is in  $H^\infty$  with bound uniform in  $t_0$ , linear in  $\|\eta\|_{\Psi_{-\delta}^\delta}$ , and clearly independent of  $r$ . Furthermore, the function  $\theta : z \mapsto z^{\sigma-\delta}\varphi(z)$  is in  $\Psi_{\sigma+\tau}^0$ , and so we can write

$$r^{-\delta}\tilde{S}_{t_0,r} = \gamma(t_0)(A)\theta_r(A)$$

where  $\gamma(t)$  is uniformly in  $H^\infty$  and  $\theta \in \Psi_{\sigma+\tau}^0$ . Theorem 5.2.8 then implies that the family  $(\tilde{S}_{t_0,r})_{r \in (t_0, \infty)}$  satisfies off-diagonal estimates of order  $\sigma + \tau$  uniformly in  $t_0 > 0$ , with implicit constants linear in  $\|\eta\|_\infty$ .

Likewise, if  $r_0 \leq t$  we can write

$$\begin{aligned} t^{-\delta} \tilde{S}_{t,r_0} &= t^{-\delta} (t/r_0)^{\tau+\delta} [\psi_t(z) \eta(z) \varphi_{r_0}(z)](A) \\ &= [(r_0 z)^{-(\tau+\delta)} \varphi_{r_0}(z) \eta^\delta(z) (tz)^\tau \psi_t(z)](A) \end{aligned}$$

and proceed in the same way, the consequence being that  $(t^{-\delta} \tilde{S}_{t,r_0})_{t \in (r_0, \infty)}$  satisfies off-diagonal estimates of order  $\sigma + \tau$  uniformly in  $r_0 > 0$ , with implicit constants linear in  $\|\eta\|_{\Psi_{-\delta}^\delta}$ .  $\square$

**Lemma 5.2.19.** *Fix  $s, \delta \in \mathbb{R}$ . Suppose  $\psi \in \Psi_{(s+\delta)_+}^{-(s+\delta)+}$ ,  $\varphi \in \Psi_{-s+}^{s+}$ , and  $\eta \in \Psi_{-\delta}^\delta$ . Then the operator  $S \sim ((\psi_t \eta \varphi_r)(A))_{t,r>0}$  extends to a bounded operator  $L_s^2 \rightarrow L_{s+\delta}^2$ .*

*Proof.* Fix  $\varepsilon > 0$  such that  $\psi \in \Psi_{\varepsilon+s+\delta}^{\varepsilon-(s+\delta)}$  and  $\varphi \in \Psi_{\varepsilon-s}^{\varepsilon+s}$ . First note that  $\psi_t \eta \varphi_r \in \Psi_+^+$ , so the operators  $S_{t,r} := (\psi_t \eta \varphi_r)(A)$  are all bounded and defined by the integral (5.22) on  $L^2$ . We will make use of Lemma 5.2.13, so we write  $r = \kappa t$  and begin by estimating

$$\begin{aligned} \|r^{-\delta} S_{t,r}\|_{\mathcal{L}(L^2(\mathbb{R}^n))} &\lesssim_{\psi, \varphi} (\kappa t)^{-\delta} \|\eta_{1/t}\|_{\Psi_{-\delta}^\delta} \int_0^\infty m_{\varepsilon+s}^{\varepsilon-s}(t\lambda) m_{\varepsilon-s}^{\varepsilon+s}(\kappa t\lambda) \frac{d\lambda}{\lambda} \\ &\leq \kappa^{-\delta} \|\eta\|_{\Psi_{-\delta}^\delta} \int_0^\infty m_{\varepsilon+s}^{\varepsilon-s}(\lambda) m_{\varepsilon-s}^{\varepsilon+s}(\kappa\lambda) \frac{d\lambda}{\lambda}. \end{aligned} \quad (5.45)$$

using Lemma 5.2.3 to eliminate the powers of  $t$  in (5.45). If  $\kappa \leq 1$ , we have

$$\begin{aligned} &\kappa^{-\delta} \int_0^\infty m_{\varepsilon+s}^{\varepsilon-s}(\lambda) m_{\varepsilon-s}^{\varepsilon+s}(\kappa\lambda) \frac{d\lambda}{\lambda} \\ &= \kappa^{-\delta} \left( \kappa^{\varepsilon-s} \int_0^1 \lambda^{2\varepsilon} \frac{d\lambda}{\lambda} + \kappa^{\varepsilon-s} \int_1^{1/\kappa} \frac{d\lambda}{\lambda} + \kappa^{-\varepsilon-s} \int_{1/\kappa}^\infty \lambda^{-2\varepsilon} \frac{d\lambda}{\lambda} \right) \\ &\lesssim \kappa^{\varepsilon-s-\delta} (2 + \log(1/\kappa)). \end{aligned}$$

If  $\kappa \geq 1$ , then by the same argument we have

$$\kappa^{-\delta} \int_0^\infty m_{\varepsilon+s}^{\varepsilon-s}(\lambda) m_{\varepsilon-s}^{\varepsilon+s}(\kappa\lambda) \frac{d\lambda}{\lambda} \lesssim \kappa^{-\varepsilon-s-\delta} (2 + \log(\kappa)).$$

Since the function

$$\gamma(\kappa) := \begin{cases} \kappa^\varepsilon (2 + \log(1/\kappa)) & (\kappa \leq 1) \\ \kappa^{-\varepsilon} (2 + \log(\kappa)) & (\kappa \geq 1) \end{cases}$$

is in  $L^1(\mathbb{R}_+)$ , Lemma 5.2.13 completes the proof.  $\square$

The following theorem is the basis of Chapter 6. From the viewpoint of applications, the important part of this theorem is the decay condition on  $\psi$  at 0. We would particularly like to take  $\psi = \text{sgp} \in \Psi_0^\infty$  and  $\delta = 0$ , which is possible provided that  $i(\mathbf{p}) \leq 2$  and  $\theta(\mathbf{p}) < 0$ .

**Theorem 5.2.20** (Boundedness of contraction/extension compositions). *Suppose  $\mathbf{p}$  is an exponent,  $\delta \in \mathbb{R}$  and  $\eta \in \Psi_{-\delta}^\delta$ . Suppose that either*

- $i(\mathbf{p}) \leq 2$  and

$$\psi \in \Psi_{(\theta(\mathbf{p})+\delta)+}^{(-(\theta(\mathbf{p})+\delta)+n|\frac{1}{2}-j(\mathbf{p})|)+} \cap H^\infty \quad \text{and} \quad \varphi \in \Psi_{(-\theta(\mathbf{p})+n|\frac{1}{2}-j(\mathbf{p})|)+}^{\theta(\mathbf{p})+} \cap \Psi_+^+, \quad (5.46)$$

or

- $i(\mathbf{p}) \geq 2$  and

$$\psi \in \Psi_{(\theta(\mathbf{p})+n|\frac{1}{2}-j(\mathbf{p})|)+}^{(-\theta(\mathbf{p})+)+} \cap \Psi_+^+ \quad \text{and} \quad \varphi \in \Psi_{(-\theta(\mathbf{p})+\delta)+}^{(\theta(\mathbf{p})-\delta+n|\frac{1}{2}-j(\mathbf{p})|)+} \cap \Psi_+^+.$$

then  $\mathbb{Q}_{\psi,A}\eta(A)\mathbb{S}_{\varphi,A}$  extends to a bounded operator  $X^{\mathbf{p}} \rightarrow X^{\mathbf{p}+\delta}$  (by duality when  $i(\mathbf{p}) = \infty$ ), with bounds linear in  $\|\eta\|_{\Psi_{-\delta}^\delta}$ .

*Proof.* We will only prove the result for tent spaces. The  $Z$ -space result can be deduced by real interpolation, or alternatively it can be proven directly via the dyadic characterisation of Proposition 5.1.19. Furthermore, the result for  $i(\mathbf{p}) \geq 2$  follows from the result for  $i(\mathbf{p}) \leq 2$  by duality, so we need only prove the result for  $i(\mathbf{p}) \leq 2$ . Note that (5.43) and the assumptions on  $\psi$  and  $\varphi$  imply that  $\mathbb{Q}_{\psi,A}\eta(A)\mathbb{S}_{\varphi,A}$  contains the integral operator with kernel  $((\psi_t\eta\varphi_\tau)(A))_{t,\tau>0}$ , so it suffices to work with this operator. Furthermore, the assumptions (5.46) and (5.2.19) imply that this operator is bounded from  $T_{\theta(\mathbf{p})}^2$  to  $T_{\theta(\mathbf{p})+\delta}^2$ , which yields the result for  $i(\mathbf{p}) = 2$ .

**Step 1:**  $i(\mathbf{p}) \leq 1$ . The assumptions (5.46) imply that there exists  $\varepsilon > 0$  such that

$$\psi \in \Psi_{\sigma+\varepsilon}^{\tau+\varepsilon} \quad \text{and} \quad \varphi \in \Psi_{(\tau+\varepsilon)+\delta}^{(\sigma+\varepsilon)-\delta},$$

where  $\sigma := \theta(\mathbf{p}) + \delta$  and  $\tau := -\theta(\mathbf{p} + \delta) + n|(1/2) - j(\mathbf{p})|$ . Therefore by Lemma 5.2.18, the operator families  $(t^{-\delta}\tilde{S}_{t,r_0})_{t \in (r_0, \infty)}$  and  $(r^{-\delta}\tilde{S}_{t_0,r})_{r \in (t_0, \infty)}$ , where  $\tilde{S}_{t,r}$  is defined as in (5.44), satisfy off-diagonal estimates of order  $n|(1/2) - j(\mathbf{p})| + 2\varepsilon$ . Theorem 5.2.14 then applies with  $a = \sigma + \varepsilon$ ,  $b = \tau + \varepsilon + \delta$ , and  $M = 2\varepsilon + n|(1/2) - j(\mathbf{p})|$ , and we can conclude that  $\mathbb{Q}_{\psi,A}\eta(A)\mathbb{S}_{\varphi,A}$  is bounded from  $T^{\mathbf{p}}$  to  $T^{\mathbf{p}+\delta}$ .

**Step 2:**  $i(\mathbf{p}) \in (1, 2)$ . The following argument originates from the thesis of Stahlhut [84, Step 4, proof of Lemma 3.2.6]. For  $\lambda \in \mathbb{C}$ , define functions  $\psi^\lambda$  and  $\varphi^\lambda$  by

$$\psi^\lambda(z) := \left( \frac{[z]}{1 + [z]} \right)^\lambda \psi(z), \quad \varphi^\lambda(z) := \left( \frac{1}{1 + [z]} \right)^\lambda \varphi(z).$$

If  $\operatorname{Re} \lambda \geq n(1 - j(\mathbf{p}))$ , then Step 1 applies with exponent  $(1, \theta(\mathbf{p}))$ , and we find that  $\mathbb{Q}_{\psi^\lambda, A} \eta(A) \mathbb{S}_{\varphi^\lambda, A}$  is bounded from  $T_{\theta(\mathbf{p})}^1$  to  $T_{\theta(\mathbf{p})+\delta}^1$ . Furthermore, if  $\operatorname{Re} \lambda \geq -n|(1/2) - j(\mathbf{p})|$ , then the discussion of the first paragraph of the proof applies, and we find that  $\mathbb{Q}_{\psi^\lambda, A} \eta(A) \mathbb{S}_{\varphi^\lambda, A}$  is bounded from  $T_{\theta(\mathbf{p})}^2$  to  $T_{\theta(\mathbf{p})+\delta}^2$ . By Stein interpolation in tent spaces (see [10, Proof of Lemma 3.4]), when  $\operatorname{Re} \lambda = 0$ , we have that  $\mathbb{Q}_{\psi^\lambda, A} \eta(A) \mathbb{S}_{\varphi^\lambda, A}$  is bounded from  $T_{\theta(\mathbf{p})}^p$  to  $T_{\theta(\mathbf{p})+\delta}^p$  when  $p \in (1, 2)$  and  $\theta \in (0, 1)$  satisfy

$$\frac{1}{p} = (1 - \theta) + \frac{\theta}{2}, \quad 0 = (1 - \theta)(1 - j(\mathbf{p})) + \theta \left( \frac{1}{2} - j(\mathbf{p}) \right).$$

This occurs when  $p = i(\mathbf{p})$ . Applying this with  $\lambda = 0$  yields boundedness of  $\mathbb{Q}_{\psi, A} \eta(A) \mathbb{S}_{\varphi, A}$  from  $T^{\mathbf{p}}$  to  $T^{\mathbf{p}+\delta}$ .  $\square$

Finally, we shall discuss an abstract form of the Calderón reproducing formula, which is ubiquitous in the study of abstract Hardy spaces, and which will play an important role in what follows.

Whenever  $\psi \in \Psi_+^+$  and  $\varphi \in H^\infty$ , we can define a bounded holomorphic function

$$\Phi_{\psi, \varphi}(z) := \int_0^\infty \psi_t(z) \varphi_t(z) \frac{dt}{t}, \quad z \in S_\mu.$$

This integral converges absolutely because  $\psi\varphi \in \Psi_+^+$ . It is not hard to show that

$$\mathbb{S}_{\psi, A} \mathbb{Q}_{\varphi, A} = \Phi_{\psi, \varphi}(A)$$

as operators on  $\overline{\mathcal{R}(A)}$ .

In [16, Proposition 4.2] it is shown that if  $\varphi \in H^\infty$  is nondegenerate, then there exists  $\psi \in \Psi_\infty^\infty$  such that  $\Phi_{\varphi, \psi} \equiv 1$ . This implies the following abstract Calderón reproducing formula.

**Theorem 5.2.21.** *Suppose  $\varphi \in H^\infty$  is nondegenerate. Then there exists  $\psi \in \Psi_\infty^\infty$  such that*

$$\mathbb{S}_{\psi, A} \mathbb{Q}_{\varphi, A} = I_{\overline{\mathcal{R}(A)}} \tag{5.47}$$

*as operators on  $\overline{\mathcal{R}(A)}$ . Furthermore, if  $\varphi \in \Psi_+^+$ , then the operator  $\mathbb{Q}_{\psi, A} \mathbb{S}_{\varphi, A}$  is a projection from  $L^2(\mathbb{R}_+^{1+n})$  onto  $\mathbb{Q}_{\psi, A} \overline{\mathcal{R}(A)}$ .*

We refer to a pair  $(\varphi, \psi)$ , with  $\varphi \in H^\infty$ ,  $\psi \in \Psi_+^+$  and satisfying (5.47), as *Calderón siblings*.

Here is a simple example of the use of the abstract Calderón reproducing formula.

**Corollary 5.2.22.** *Suppose  $\varphi \in H^\infty$  is nondegenerate. Then the extension operator  $\mathbb{Q}_{\varphi,A}: \overline{\mathcal{R}(A)} \rightarrow L^2(\mathbb{R}_+ : \overline{\mathcal{R}(A)})$  is injective.*

*Proof.* Let  $\psi \in \Psi_+^+$  be a Calderón sibling of  $\varphi$ , and suppose  $f \in \overline{\mathcal{R}(A)}$  with  $\mathbb{Q}_{\varphi,A}f = 0$ . Then by (5.47) we have

$$f = \mathbb{S}_{\psi,A} \mathbb{Q}_{\varphi,A}f = 0,$$

and so  $\mathbb{Q}_{\varphi,A}$  is injective. □





# Chapter 6

## Adapted function spaces

### 6.1 Adapted Hardy–Sobolev and Besov spaces

Throughout this section we will fix an operator  $A$  satisfying the standard assumptions (see Definition 5.2.11). As in the previous chapter, we will implicitly work with  $\mathbb{C}^N$ -valued functions without referencing this in the notation.

#### 6.1.1 Initial definitions, equivalent norms, and duality

The adapted Hardy–Sobolev and Besov spaces are, defined, roughly speaking, by measuring extensions by  $\mathbb{Q}_{\psi,A}$  in tent spaces and  $Z$ -spaces respectively. We will soon show that the resulting function space is independent of  $\psi$  for  $\psi$  with sufficient decay at 0 and  $\infty$  depending on  $\mathbf{p}$ .

**Definition 6.1.1.** Let  $\psi \in H^\infty$  and let  $\mathbf{p}$  be an exponent. We define the sets

$$\begin{aligned}\mathbb{H}_{\psi,A}^{\mathbf{p}} &:= \{f \in \overline{\mathcal{R}(A)} : \mathbb{Q}_{\psi,A}f \in T^{\mathbf{p}}\}, \\ \mathbb{B}_{\psi,A}^{\mathbf{p}} &:= \{f \in \overline{\mathcal{R}(A)} : \mathbb{Q}_{\psi,A}f \in Z^{\mathbf{p}}\},\end{aligned}$$

equipped with quasinorms<sup>1</sup>

$$\begin{aligned}\|f\|_{\mathbb{H}_{\psi,A}^{\mathbf{p}}} &:= \|\mathbb{Q}_{\psi,A}f\|_{T^{\mathbf{p}}}, \\ \|f\|_{\mathbb{B}_{\psi,A}^{\mathbf{p}}} &:= \|\mathbb{Q}_{\psi,A}f\|_{Z^{\mathbf{p}}}.\end{aligned}$$

We call these spaces *pre-Hardy–Sobolev* and *pre-Besov spaces associated with  $A$*  (respectively), and we call  $\psi$  an *auxiliary function*.

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<sup>1</sup>These are shown to be quasinorms in Proposition 6.1.2. Of course, they are actual norms when  $i(\mathbf{p}) \geq 1$ .

Generally we will want to refer to the pre-Hardy–Sobolev and pre-Besov spaces simultaneously. In this case we will write

$$\mathbb{X}_{\psi,A}^{\mathbf{p}} := \{f \in \overline{\mathcal{R}(A)} : \mathbb{Q}_{\psi,A}f \in X^{\mathbf{p}}\},$$

where the pair  $(X, \mathbb{X})$  is either  $(T, \mathbb{H})$  or  $(Z, \mathbb{B})$ . This follows the convention initiated in Subsection 5.1.4.

**Proposition 6.1.2.** *Let  $\psi \in H^\infty$  and let  $\mathbf{p}$  be an exponent. Then  $\|\cdot\|_{\mathbb{X}_{\psi,A}^{\mathbf{p}}}$  is a quasinorm on  $\mathbb{X}_{\psi,A}^{\mathbf{p}}$ .*

*Proof.* The only quasinorm property which does not follow directly from linearity of  $\mathbb{Q}_{\psi,A}$  and the corresponding quasinorm properties of  $X^{\mathbf{p}}$  is positive definiteness. To show this, suppose  $f \in \mathbb{X}_{\psi,A}^{\mathbf{p}}$  and  $\|f\|_{\mathbb{X}_{\psi,A}^{\mathbf{p}}} = 0$ . Then we have  $\mathbb{Q}_{\psi,A}f = 0$  in  $X^{\mathbf{p}}$ , and hence also in  $L^2(\mathbb{R}_+ : \overline{\mathcal{R}(A)})$ . By injectivity of  $\mathbb{Q}_{\psi,A} : \overline{\mathcal{R}(A)} \rightarrow L^2(\mathbb{R}_+ : \overline{\mathcal{R}(A)})$  (Corollary 5.2.22), we conclude that  $f = 0$ .  $\square$

The following proposition quantifies the amount of decay needed on the auxiliary function  $\psi$  in order to ensure that the  $\mathbb{X}_{\psi,A}^{\mathbf{p}}$  quasinorm is equivalent to the  $\mathbb{X}_{\varphi,A}^{\mathbf{p}}$  quasinorm whenever  $\varphi$  has decay of arbitrarily high order at 0 and  $\infty$ .

**Proposition 6.1.3** (Independence on auxiliary function). *Let  $\varphi \in \Psi_\infty^\infty$  and  $\psi \in H^\infty$  be nondegenerate, let  $\mathbf{p}$  be an exponent, and suppose that either*

- $i(\mathbf{p}) \leq 2$  and  $\psi \in \Psi_{\theta(\mathbf{p})+}^{(-\theta(\mathbf{p})+n|\frac{1}{2}-j(\mathbf{p})|)+}$ , or
- $i(\mathbf{p}) \geq 2$  and  $\psi \in \Psi_{(\theta(\mathbf{p})+n|\frac{1}{2}-j(\mathbf{p})|)+}^{-\theta(\mathbf{p})+} \cap \Psi_+^+$ .

*Then we have  $\mathbb{X}_{\psi,A}^{\mathbf{p}} = \mathbb{X}_{\varphi,A}^{\mathbf{p}}$  with equivalent quasinorms.*

*Proof.* First, let  $\nu \in \Psi_\infty^\infty$  be a Calderón sibling of  $\psi$ . Then for  $f \in \mathbb{X}_{\psi,A}^{\mathbf{p}}$  we have

$$\begin{aligned} \|f\|_{\mathbb{X}_{\psi,A}^{\mathbf{p}}} &= \|\mathbb{Q}_{\varphi,A}f\|_{X^{\mathbf{p}}} \\ &= \|\mathbb{Q}_{\varphi,A}\mathbb{S}_{\nu,A}\mathbb{Q}_{\psi,A}f\|_{X^{\mathbf{p}}} \\ &\lesssim \|\mathbb{Q}_{\psi,A}f\|_{X^{\mathbf{p}}} \\ &= \|f\|_{\mathbb{X}_{\psi,A}^{\mathbf{p}}}, \end{aligned} \tag{6.1}$$

where (6.1) follows from Theorem 5.2.20, by the standard assumptions along with  $\varphi, \nu \in \Psi_\infty^\infty$ .

Now let  $\nu \in \Psi_\infty^\infty$  be a Calderón sibling of  $\varphi$ . Then we can repeat the previous argument with the roles of  $\varphi$  and  $\psi$  reversed, using the additional assumptions on  $\psi$  to apply Theorem 5.2.20. This leads to the reverse estimate

$$\|f\|_{\mathbb{X}_{\psi,A}^{\mathbf{p}}} \lesssim \|f\|_{\mathbb{X}_{\varphi,A}^{\mathbf{p}}}$$

which completes the proof.  $\square$

**Definition 6.1.4.** For an exponent  $\mathbf{p}$ , we define the spaces

$$\mathbb{X}_A^{\mathbf{p}} := \mathbb{X}_{\psi, A}^{\mathbf{p}},$$

where any auxiliary function  $\psi \in \Psi_{\infty}^{\infty}$  may be used to define the space and its corresponding quasinorm. We also define  $\Psi(\mathbb{X}_A^{\mathbf{p}})$  to be the set of all nondegenerate  $\varphi \in H^{\infty}$  such that  $\mathbb{X}_{\varphi, A}^{\mathbf{p}} = \mathbb{X}_A^{\mathbf{p}}$  with equivalent quasinorms.

With this notation at hand, Proposition 6.1.3 tells us that

$$\begin{aligned} \Psi_{\theta(\mathbf{p})+}^{(-\theta(\mathbf{p})+n|\frac{1}{2}-j(\mathbf{p})|)+} \cap H^{\infty} &\subset \Psi(\mathbb{X}_A^{\mathbf{p}}) & (i(\mathbf{p}) \leq 2), \\ \Psi_{(\theta(\mathbf{p})+n|\frac{1}{2}-j(\mathbf{p})|)+}^{-\theta(\mathbf{p})+} \cap \Psi_+^+ &\subset \Psi(\mathbb{X}_A^{\mathbf{p}}) & (i(\mathbf{p}) > 2). \end{aligned}$$

Recall that the positive and negative spectral subspaces

$$\overline{\mathcal{R}(A)}^{\pm} := \chi^{\pm}(A) \overline{\mathcal{R}(A)}$$

were defined and discussed in Section 5.2.1. These can be used to define corresponding positive and negative spectral subspaces of  $\mathbb{X}_A^{\mathbf{p}}$ .

**Definition 6.1.5.** Let  $\mathbf{p}$  be an exponent. Then we define the positive and negative pre-Hardy–Sobolev and Besov spaces by

$$\mathbb{X}_A^{\mathbf{p}, \pm} := \mathbb{X}_A^{\mathbf{p}} \cap \overline{\mathcal{R}(A)}^{\pm},$$

equipped with any of the equivalent  $\mathbb{X}_A^{\mathbf{p}}$  quasinorms. Often we will just refer to these as the *spectral subspaces*.

In Corollary 6.1.13 we will characterise the positive and negative spaces  $\mathbb{X}_A^{\mathbf{p}, \pm}$  as images of the spectral projections  $\chi^{\pm}(A)$ .

The spaces  $\mathbb{X}_A^{\mathbf{p}}$  may also be characterised in terms of the contraction maps  $\mathbb{S}_{\psi, A}$  for any  $\psi \in \Psi_+^+$ . Recall that  $X^2 = T^2 = Z^2 = L^2(\mathbb{R}_+^{1+n})$ .

**Proposition 6.1.6** (Characterisation by contraction maps). *Let  $\mathbf{p}$  be an exponent and let  $\psi \in \Psi_+^+$  be nondegenerate. Then we have*

$$\mathbb{X}_A^{\mathbf{p}} = \mathbb{S}_{\psi, A} \left( X^2 \cap X^{\mathbf{p}} \right), \tag{6.2}$$

and the mapping

$$f \mapsto \inf \{ \|F\|_{X^{\mathbf{p}}} : F \in X^2 \cap X^{\mathbf{p}}, \mathbb{S}_{\psi, A} F = f \}$$

is an equivalent quasinorm on  $\mathbb{X}_A^{\mathbf{p}}$ .

*Proof.* Fix a Calderón sibling  $\varphi \in \Psi^\infty$  of  $\psi$ . First we will show the equality (6.2). Suppose  $f \in \mathbb{X}_A^{\mathbf{p}}$ . Then  $\mathbb{Q}_{\varphi,A}f \in X^2 \cap X^{\mathbf{p}}$ , and by Theorem 5.47 we have  $f = \mathbb{S}_{\psi,A}(\mathbb{Q}_{\varphi,A}f)$ . Conversely, suppose that  $f = \mathbb{S}_{\psi,A}F$  for some  $F \in X^2 \cap X^{\mathbf{p}}$ . Then  $f \in \overline{\mathcal{R}(A)}$ , and Theorem 5.47 implies that  $\mathbb{Q}_{\varphi,A}f = F \in X^{\mathbf{p}}$ , which shows that  $f \in \mathbb{X}_A^{\mathbf{p}}$ . This proves (6.2).

Now prove the quasinorm equivalence. Suppose  $f \in \mathbb{X}_A^{\mathbf{p}}$ . Then  $f = \mathbb{S}_{\psi,A}\mathbb{Q}_{\varphi,A}f$  with  $\mathbb{Q}_{\varphi,A}f \in X^2 \cap X^{\mathbf{p}}$ , and so

$$\inf\{\|F\|_{X^{\mathbf{p}}} : F \in X^2 \cap X^{\mathbf{p}}, \mathbb{S}_{\psi,A}F = f\} \leq \|\mathbb{Q}_{\varphi,A}f\|_{X^{\mathbf{p}}} \simeq \|f\|_{\mathbb{X}_A^{\mathbf{p}}}.$$

Conversely, suppose  $F \in X^2 \cap X^{\mathbf{p}}$  and  $\mathbb{S}_{\psi,A}F = f$ . Then

$$\begin{aligned} \|F\|_{X^{\mathbf{p}}} &\gtrsim \|\mathbb{Q}_{\varphi,A}\mathbb{S}_{\psi,A}F\|_{X^{\mathbf{p}}} \\ &= \|\mathbb{Q}_{\varphi,A}f\|_{X^{\mathbf{p}}} \\ &\simeq \|f\|_{\mathbb{X}_A^{\mathbf{p}}}, \end{aligned}$$

completing the proof.  $\square$

**Corollary 6.1.7** (Density of intersections). *Let  $\mathbf{p}$  and  $\mathbf{q}$  be exponents, and suppose  $\mathbb{X}_1, \mathbb{X}_2 \in \{\mathbb{H}, \mathbb{B}\}$ . If  $\mathbf{p}$  is finite then  $(\mathbb{X}_1)_A^{\mathbf{p}} \cap (\mathbb{X}_2)_A^{\mathbf{q}}$  is dense in  $(\mathbb{X}_1)_A^{\mathbf{p}}$ . Otherwise,  $(\mathbb{X}_1)_A^{\mathbf{p}} \cap (\mathbb{X}_2)_A^{\mathbf{q}}$  is weak-star dense in  $(\mathbb{X}_1)_A^{\mathbf{p}}$ .*

*Proof.* We will suppose that  $\mathbf{p}$  is finite; the same argument works for infinite  $\mathbf{p}$ , replacing limits with weak-star limits and norms with appropriate duality pairings.

Suppose  $f \in (\mathbb{X}_1)_A^{\mathbf{p}}$ , and fix  $\psi \in \Psi_+^+$ . By Proposition 6.1.6 we can write  $f = \mathbb{S}_{\psi,A}F$  for some  $F \in T^2 \cap (X_1)^{\mathbf{p}}$ , and by Proposition 5.1.35 we can write  $F = \lim_{k \rightarrow \infty} F_k$  (limit in  $(X_1)^{\mathbf{p}}$ ) for some sequence  $(F_k)_{k \in \mathbb{N}}$  in  $T^2 \cap (X_1)^{\mathbf{p}} \cap (X_2)^{\mathbf{q}}$ . For all  $k \in \mathbb{N}$  define

$$f_k := \mathbb{S}_{\psi,A}F_k \in (\mathbb{X}_1)_A^{\mathbf{p}} \cap (\mathbb{X}_2)_A^{\mathbf{q}}.$$

Then we have, again using Proposition 6.1.6,

$$\lim_{k \rightarrow \infty} \|f - f_k\|_{(\mathbb{X}_1)_A^{\mathbf{p}}} \lesssim \lim_{k \rightarrow \infty} \|F - F_k\|_{(X_1)^{\mathbf{p}}} = 0.$$

This proves the claimed density.  $\square$

The pre-Hardy–Sobolev and pre-Besov spaces inherit a duality pairing from  $\overline{\mathcal{R}(A)} \subset L^2(\mathbb{R}^n)$ . However, we cannot say that  $\mathbb{X}_A^{\mathbf{p}'}$  is the dual of  $\mathbb{X}_A^{\mathbf{p}}$ , because in general these spaces are incomplete, while the dual of a quasinormed space is always complete. We will deal with completions in Subsection 6.1.3.

**Proposition 6.1.8** (Duality estimate). *Let  $\mathbf{p}$  be an exponent. Then for all  $f \in \mathbb{X}_A^{\mathbf{p}}$  and  $g \in \mathbb{X}_{A^*}^{\mathbf{p}'}$  we have*

$$|\langle f, g \rangle| \lesssim \|f\|_{\mathbb{X}_A^{\mathbf{p}}} \|g\|_{\mathbb{X}_{A^*}^{\mathbf{p}'}} , \quad (6.3)$$

where  $\langle \cdot, \cdot \rangle$  is the inner product on  $L^2(\mathbb{R}^n)$ .

*Proof.* Let  $\varphi, \psi \in \Psi_\infty^\infty$  be nondegenerate and suppose  $\varepsilon > 0$ . By Proposition 6.1.6 there exist  $F \in X^2 \cap X^{\mathbf{p}}$  and  $G \in X^2 \cap X^{\mathbf{p}'}$  such that  $\mathbb{S}_{\varphi, A} F = f$  and  $\mathbb{S}_{\psi, A^*} G = g$ , with

$$\|F\|_{X^{\mathbf{p}}} \lesssim (1 + \varepsilon) \|f\|_{\mathbb{X}_A^{\mathbf{p}}} \quad \text{and} \quad \|G\|_{X^{\mathbf{p}'}} \lesssim (1 + \varepsilon) \|g\|_{\mathbb{X}_{A^*}^{\mathbf{p}'}} .$$

Since  $\mathbb{S}_{\varphi, A}^* = \mathbb{Q}_{\tilde{\varphi}, A^*}$ , and using that the  $L^2(\mathbb{R}_+^{1+n})$  inner product yields a duality pairing for tent and  $Z$ -spaces, we thus have

$$\begin{aligned} |\langle f, g \rangle| &= |\langle F, \mathbb{Q}_{\tilde{\varphi}, A^*} \mathbb{S}_{\psi, A^*} G \rangle_{L^2(\mathbb{R}_+^{1+n})}| \\ &\lesssim \|F\|_{X^{\mathbf{p}}} \|\mathbb{Q}_{\tilde{\varphi}, A^*} \mathbb{S}_{\psi, A^*} G\|_{X^{\mathbf{p}'}} \\ &\lesssim \|F\|_{X^{\mathbf{p}}} \|G\|_{X^{\mathbf{p}'}} \\ &\lesssim (1 + \varepsilon)^2 \|f\|_{\mathbb{X}_A^{\mathbf{p}}} \|g\|_{\mathbb{X}_{A^*}^{\mathbf{p}'}} , \end{aligned} \quad (6.4)$$

where (6.4) follows from Theorem 5.2.20. Since  $\varepsilon > 0$  was arbitrary we obtain (6.3).  $\square$

The tent space and  $Z$ -space embeddings of Section 5.1 immediately yield corresponding embeddings of the pre-Hardy–Sobolev and pre-Besov spaces.

**Proposition 6.1.9** (Mixed embeddings). *Let  $\mathbf{p}$  and  $\mathbf{q}$  be exponents with  $\mathbf{p} \neq \mathbf{q}$  and  $\mathbf{p} \hookrightarrow \mathbf{q}$ . Then we have the continuous embedding*

$$(\mathbb{X}_0)_A^{\mathbf{p}} \hookrightarrow (\mathbb{X}_1)_A^{\mathbf{q}},$$

where  $\mathbb{X}_0, \mathbb{X}_1 \in \{\mathbb{H}, \mathbb{B}\}$ , and the corresponding embedding holds for positive and negative versions.

*Proof.* This follows directly from the definition of the spaces  $\mathbb{X}_A^{\mathbf{p}}$  and from Theorem 5.1.33. The spectral subspace versions follow by intersecting with  $\overline{\mathcal{R}(A)}^\pm$ .  $\square$

*Remark 6.1.10.* For  $\mathbf{p} = (p, s)$  we will sometimes write  $\mathbb{X}_A^{\mathbf{p}} = \mathbb{X}_{s, A}^p$ , and for  $\mathbf{p} = (\infty, s; \alpha)$  we may write  $\mathbb{X}_A^{\mathbf{p}} = \mathbb{X}_{s; \alpha, A}^p$ . This notation is a bit heavy, so we avoid it whenever possible, except in the case of  $\mathbb{X}_{0, A}^2$ , which we can simply abbreviate as  $\mathbb{X}_A^2$  (as is standard).

### 6.1.2 Mapping properties of holomorphic functional calculus

In the same way that we proved independence on auxiliary functions in the previous section, we can prove various mapping properties (including boundedness) of the holomorphic functional calculus between pre-Hardy–Sobolev and pre-Besov spaces.

The first result says heuristically that an operator of homogeneity  $\delta$  decreases regularity by  $\delta$ .

**Theorem 6.1.11.** *Let  $\mathbf{p}$  be an exponent and  $\delta \in \mathbb{R}$ . Suppose  $\eta \in \Psi_{-\delta}^\delta$ . Then the operator  $\eta(A)$  maps  $\mathcal{D}(\eta(A)) \cap \mathbb{X}_A^{\mathbf{p}}$  into  $\mathbb{X}_A^{\mathbf{p}+\delta}$ , and the quasinorm estimate*

$$\|\eta(A)f\|_{\mathbb{X}_A^{\mathbf{p}+\delta}} \lesssim \|\eta\|_{\Psi_{-\delta}^\delta} \|f\|_{\mathbb{X}_A^{\mathbf{p}}}$$

*holds for all  $f \in \mathcal{D}(\eta(A)) \cap \mathbb{X}_A^{\mathbf{p}}$ . The same results hold for spectral subspaces.*

*Proof.* Let  $\varphi, \psi \in \Psi_\infty^\infty$  and let  $\nu \in \Psi_\infty^\infty$  be a Calderón sibling of  $\psi$ . Then for all  $f \in \mathcal{D}(\eta(A)) \cap \mathbb{X}_A^{\mathbf{p}}$  we have

$$\begin{aligned} \|\eta(A)f\|_{\mathbb{X}_A^{\mathbf{p}+\delta}} &\simeq \|\mathbb{Q}_{\varphi,A}\eta(A)\mathbb{S}_{\nu,A}\mathbb{Q}_{\psi,A}f\|_{X^{\mathbf{p}+\delta}} \\ &\lesssim \|\eta\|_{\Psi_{-\delta}^\delta} \|\mathbb{Q}_{\psi}f\|_{X^{\mathbf{p}}} \\ &\simeq \|\eta\|_{\Psi_{-\delta}^\delta} \|f\|_{\mathbb{X}_A^{\mathbf{p}}} \end{aligned} \tag{6.5}$$

where (6.5) follows from Theorem 5.2.20. To incorporate spectral subspaces in this argument, write  $f \in \mathcal{D}(\eta(A)) \cap \mathbb{X}_A^{\mathbf{p},\pm}$  as  $f = \chi^\pm(A)f$  and

$$\eta(A)f = \eta(A)\chi^\pm(A)f = \chi^\pm(A)\eta(A)f,$$

and note that this shows that  $\eta(A)$  maps  $\mathcal{D}(\eta(A)) \cap \overline{\mathcal{R}(A)}^\pm$  into  $\overline{\mathcal{R}(A)}^\pm$ .  $\square$

Because the spaces  $\mathbb{X}_A^{\mathbf{p}}$  may be incomplete, we cannot extend the operators  $\eta(A)$  by boundedness without introducing completions. This is done in Subsection 6.1.3. Of course, when  $\eta \in H^\infty$  we have  $\mathcal{D}(\eta(A)) = \overline{\mathcal{R}(A)}$ , and so we obtain bounded holomorphic functional calculus in the following sense.

**Corollary 6.1.12.** *Let  $\mathbf{p}$  be an exponent and  $\eta \in H^\infty$ . Then the operator  $\eta(A)$  is bounded on  $\mathbb{X}_A^{\mathbf{p}}$ , with*

$$\|\eta(A)f\|_{\mathbb{X}_A^{\mathbf{p}}} \lesssim \|\eta\|_\infty \|f\|_{\mathbb{X}_A^{\mathbf{p}}}$$

*for all  $f \in \mathbb{X}_A^{\mathbf{p}}$ , and likewise for spectral subspaces.*

This allows us to characterise the positive and negative subspaces  $\mathbb{X}_A^{\mathbf{p},\pm}$  as images of spectral projections.

**Corollary 6.1.13.** *Let  $\mathbf{p}$  be an exponent. Then we have*

$$\mathbb{X}_A^{\mathbf{p},\pm} = \chi^\pm(A)\mathbb{X}_A^{\mathbf{p}}.$$

*Proof.* If  $f \in \mathbb{X}_A^{\mathbf{p},\pm}$  then by definition we have  $f = \chi^\pm(A)f \in \chi^\pm(A)\mathbb{X}_A^{\mathbf{p}}$ . Conversely, if  $f \in \chi^\pm(A)\mathbb{X}_A^{\mathbf{p}}$ , then  $f$  is in  $\overline{\mathcal{R}(A)}^\pm$ , and by Corollary 6.1.12 we have  $f \in \mathbb{X}_A^{\mathbf{p}}$ . Therefore  $f \in \overline{\mathcal{R}(A)}^\pm \cap \mathbb{X}_A^{\mathbf{p}} = \mathbb{X}_A^{\mathbf{p},\pm}$ .  $\square$

The power functions  $A^\lambda := [z \mapsto z^\lambda](A)$  for  $\lambda \in \mathbb{R} \setminus \{0\}$  (see Section 5.2.1) are generally unbounded on  $\overline{\mathcal{R}(A)}$ , but they do map between our adapted spaces with a shift in regularity (when we intersect with the domain). This is a direct consequence of Theorem 6.1.11 since  $[z \mapsto z^\lambda] \in \Psi_\lambda^{-\lambda}$ ; the norm equivalence is obtained by applying Theorem 6.1.11 with both  $\lambda$  and  $-\lambda$ .

**Corollary 6.1.14.** *Let  $\mathbf{p}$  be an exponent and  $\lambda \in \mathbb{R} \setminus \{0\}$ . Then  $A^\lambda$  maps  $\mathcal{D}(A^\lambda) \cap \mathbb{X}_A^{\mathbf{p}}$  into  $\mathbb{X}_A^{\mathbf{p}-\lambda}$  with the quasinorm estimate*

$$\|A^\lambda f\|_{\mathbb{X}_A^{\mathbf{p}-\lambda}} \simeq \|f\|_{\mathbb{X}_A^{\mathbf{p}}}$$

for all  $f \in \mathcal{D}(A^\lambda) \cap \mathbb{X}_A^{\mathbf{p}}$ .

Since the operators  $A^\lambda$  are all densely defined in  $\overline{\mathcal{R}(A)}$ , and since  $A^{\lambda_0}A^{\lambda_1} = A^{\lambda_0+\lambda_1}$  whenever this is meaningful, we have almost proven that  $A^\lambda$  is an isomorphism from  $\mathbb{X}_A^{\mathbf{p}}$  to  $\mathbb{X}_A^{\mathbf{p}-\lambda}$ . We need to extend everything by boundedness to make this rigorous. As previously mentioned, this requires the introduction of completions.

### 6.1.3 Completions and interpolation

The spaces  $\mathbb{X}_A^{\mathbf{p}}$  defined in the previous section are called *pre-Hardy–Sobolev* and *pre-Besov* spaces because, with the exception of  $\mathbb{X}_{0,A}^2 = \overline{\mathcal{R}(A)}$ , they need not be complete. One could try to solve this problem by taking arbitrary completions  $(\mathbb{X}_A^{\mathbf{p}})^c$  of  $\mathbb{X}_A^{\mathbf{p}}$  and declaring these to be the Hardy–Sobolev and Besov spaces associated with  $A$ . However, if we take this approach, then for different exponents  $\mathbf{p}_i$ , there may not exist a natural topological vector space in which the completions  $(\mathbb{X}_A^{\mathbf{p}_i})^c$  both embed.<sup>2</sup> This prevents us from discussing interpolants of these

<sup>2</sup>Stahlhut takes this approach in his thesis [84, §4.1], but his ambient space - a product space of abstract completions indexed over all exponents - is not as natural as the one we are about to propose.

completions. The impact of this problem on abstract Hardy space theory seems to have first been discussed by Auscher, McIntosh, and Morris [11]. We avoid this issue by introducing certain canonical completions within tent and  $Z$ -spaces (and hence within  $L^0(\mathbb{R}_+^{1+n})$ ). If another completion is possible - for example, within the space  $\mathcal{Z}'(\mathbb{R}^n)$  of distributions modulo polynomials, in which the classical Hardy–Sobolev and Besov spaces are embedded - then we are free to identify this with our canonical completion.

By a *completion* of a quasinormed space  $Q$  we mean a continuous injective map  $\iota: Q \rightarrow \tilde{Q}$ , where  $\tilde{Q}$  is a complete quasinormed space and  $\iota(Q)$  is dense in  $\tilde{Q}$ . By a *weak-star completion* of  $Q$ , we mean  $\iota: Q \rightarrow \tilde{Q}$  as above, where  $\tilde{Q}$  is a dual space and where  $\iota(Q)$  is weak-star dense in  $\tilde{Q}$ . Eventually we will refer to  $\tilde{Q}$  itself as the completion, with the associated inclusion being implicit.

In this section, whenever  $\mathbf{p}$  is infinite, we will interpret ‘completion’ to mean ‘weak-star completion’.

**Definition 6.1.15.** For an exponent  $\mathbf{p}$  and an auxiliary function  $\psi \in \Psi(\mathbb{X}_A^{\mathbf{p}})$ , define the *canonical completion*

$$\psi\mathbf{X}_A^{\mathbf{p}} := \overline{\mathbb{Q}_{\psi,A}\mathbb{X}_A^{\mathbf{p}}} \subset X^{\mathbf{p}}$$

and likewise

$$\psi\mathbf{X}_A^{\mathbf{p},\pm} := \overline{\mathbb{Q}_{\psi,A}\mathbb{X}_A^{\mathbf{p},\pm}} \subset \psi\mathbf{X}_A^{\mathbf{p}}$$

where the closures are taken in the  $X^{\mathbf{p}}$  quasinorm when  $\mathbf{p}$  is finite, and in the weak-star topology on  $X^{\mathbf{p}}$  when  $\mathbf{p}$  is infinite. We equip  $\psi\mathbf{X}_A^{\mathbf{p}}$  and  $\psi\mathbf{X}_A^{\mathbf{p},\pm}$  with the  $X^{\mathbf{p}}$  quasinorm, so that  $\psi\mathbf{X}_A^{\mathbf{p}}$  and  $\psi\mathbf{X}_A^{\mathbf{p},\pm}$  become quasi-Banach spaces.

**Proposition 6.1.16.** Fix  $\mathbf{p}$  and  $\psi$  as in Definition 6.1.15. Then  $\mathbb{Q}_{\psi,A}: \mathbb{X}_A^{\mathbf{p}} \rightarrow \psi\mathbf{X}_A^{\mathbf{p}}$  and  $\mathbb{Q}_{\psi,A}: \mathbb{X}_A^{\mathbf{p},\pm} \rightarrow \psi\mathbf{X}_A^{\mathbf{p},\pm}$  are completions of  $\mathbb{X}_A^{\mathbf{p}}$  and  $\mathbb{X}_A^{\mathbf{p},\pm}$ .

*Proof.* By construction the spaces  $\psi\mathbf{X}_{s,A}^{\mathbf{p}}$  and  $\psi\mathbf{X}_{s,A}^{\mathbf{p},\pm}$  are complete and contain  $\mathbb{Q}_{\psi,A}\mathbb{X}_A^{\mathbf{p}}$  and  $\mathbb{Q}_{\psi,A}\mathbb{X}_A^{\mathbf{p},\pm}$  respectively as dense subspaces (in the weak-star topology when  $\mathbf{p}$  is infinite). The map  $\mathbb{Q}_{\psi,A}: \mathbb{X}_A^{\mathbf{p}} \rightarrow \psi\mathbf{X}_A^{\mathbf{p}}$  is continuous since  $\psi \in \Psi(\mathbb{X}_A^{\mathbf{p}})$ , and injective by Corollary 5.2.22. These properties automatically continue to hold for the restrictions of  $\mathbb{Q}_{\psi,A}$  to the spectral subspaces  $\mathbb{X}_A^{\mathbf{p},\pm}$ .  $\square$

Of course, completions are always unique, and so any completion may be identified with any canonical completion. It will be useful to make this identification precise.



**Proposition 6.1.17** (Identification of completions). *Fix  $\mathbf{p}$  and  $\psi$  as in Definition 6.1.15 and suppose that  $\iota: \mathbb{X}_A^{\mathbf{p}} \rightarrow \mathbf{X}$  is a completion of  $\mathbb{X}_A^{\mathbf{p}}$ . Then the unique map  $\mathbf{Q}_{\psi,A}^{\iota}: \mathbf{X} \rightarrow X^{\mathbf{p}}$  such that the triangle*

$$\begin{array}{ccc} & \mathbf{X} & \\ \iota \uparrow & \searrow \mathbf{Q}_{\psi,A}^{\iota} & \\ \mathbb{X}_A^{\mathbf{p}} & \xrightarrow{\mathbb{Q}_{\psi,A}} & X^{\mathbf{p}} \end{array}$$

*commutes, is an isomorphism between  $\mathbf{X}$  and  $\psi \mathbf{X}_A^{\mathbf{p}}$ . Its inverse is given by the map  $\mathbf{S}_{\psi,A}^{\iota}: X^{\mathbf{p}} \rightarrow \mathbf{X}$ , which is the unique continuous extension of the map  $\iota \circ \mathbb{S}_{\nu,A}: X^2 \cap X^{\mathbf{p}} \rightarrow \mathbf{X}$  (using the weak-star topology on  $X^{\mathbf{p}}$  when  $\mathbf{p}$  is infinite) for any  $\nu \in \Psi_{\infty}^{\infty}$  which is a Calderón sibling of  $\psi$ . The same results hold if we replace all spaces with corresponding positive and negative subspaces.*

*Proof.* Since  $\mathbb{Q}_{\psi,A}: \mathbb{X}_A^{\mathbf{p}} \rightarrow \psi \mathbf{X}_A^{\mathbf{p}}$  is a completion of  $\mathbb{X}_A^{\mathbf{p}}$ , by the universal property of completions there exists a unique map  $\widetilde{\mathbf{Q}}_{\psi,A}^{\iota}: \mathbf{X} \rightarrow \psi \mathbf{X}_A^{\mathbf{p}}$  such that the triangle

$$\begin{array}{ccc} & \mathbf{X} & \\ \iota \uparrow & \searrow \widetilde{\mathbf{Q}}_{\psi,A}^{\iota} & \\ \mathbb{X}_A^{\mathbf{p}} & \xrightarrow{\mathbb{Q}_{\psi,A}} & \psi \mathbf{X}_A^{\mathbf{p}} \end{array}$$

commutes. Hence we have a commutative diagram

$$\begin{array}{ccccc} & \mathbf{X} & & & \\ & \uparrow \iota & \searrow \mathbf{Q}_{\psi,A}^{\iota} & \searrow \mathbf{Q}_{\psi,A}^{\iota} & \\ \mathbb{X}_A^{\mathbf{p}} & \xrightarrow{\mathbb{Q}_{\psi,A}} & \psi \mathbf{X}_A^{\mathbf{p}} & \xrightarrow{\text{id}} & X^{\mathbf{p}} \end{array}$$

Since

$$(\text{id} \circ \widetilde{\mathbf{Q}}_{\psi,A}^{\iota}) \circ \iota = \mathbb{Q}_{\psi,A} = \mathbf{Q}_{\psi,A}^{\iota} \circ \iota,$$

by uniqueness we must have  $\mathbf{Q}_{\psi,A}^{\iota} = \text{id} \circ \widetilde{\mathbf{Q}}_{\psi,A}^{\iota} = \widetilde{\mathbf{Q}}_{\psi,A}^{\iota}$ . Therefore it suffices to show that  $\widetilde{\mathbf{Q}}_{\psi,A}^{\iota}$  satisfies the desired properties.

To show that  $\mathbf{S}_{\psi,A}^{\iota} \widetilde{\mathbf{Q}}_{\psi,A}^{\iota} = \text{id}_{\mathbf{X}}$ , observe that we have a commutative diagram

$$\begin{array}{ccccccc} \mathbf{X} & \xrightarrow{\widetilde{\mathbf{Q}}_{\psi,A}^{\iota}} & \psi \mathbf{X}_A^{\mathbf{p}} & \hookrightarrow & X^{\mathbf{p}} & \xrightarrow{\mathbf{S}_{\psi,A}^{\iota}} & \mathbf{X} & \xrightarrow{\widetilde{\mathbf{Q}}_{\psi,A}^{\iota}} & \psi \mathbf{X}_A^{\mathbf{p}} \\ \uparrow \iota & & \uparrow \mathbb{Q}_{\psi,A} & & \uparrow \mathbb{S}_{\nu,A} & & \uparrow \iota & & \uparrow \mathbb{Q}_{\psi,A} \\ \mathbb{X}_A^{\mathbf{p}} & \xrightarrow{\mathbb{Q}_{\psi,A}} & \psi(\mathbb{X}_A^{\mathbf{p}}) & \hookrightarrow & X^2 \cap X^{\mathbf{p}} & \xrightarrow{\mathbb{S}_{\nu,A}} & \mathbb{X}_A^{\mathbf{p}} & \xrightarrow{\mathbb{Q}_{\psi,A}} & \psi(\mathbb{X}_A^{\mathbf{p}}) \\ & & & & \text{id} & & \text{id} & & \end{array} \quad (6.6)$$

Thus we have

$$\mathbf{S}_{\psi,A}^\iota \widetilde{\mathbf{Q}}_{\psi,A}^\iota \iota = \iota \circ \text{id} = \iota$$

and

$$\widetilde{\mathbf{Q}}_{\psi,A}^\iota \mathbf{S}_{\psi,A}^\iota|_{\psi(\mathbb{X}_A^\mathbf{p})} = \text{id},$$

so by uniqueness of extensions we must have that  $\widetilde{\mathbf{Q}}_{\psi,A}^\iota$  and  $\mathbf{S}_{\psi,A}^\iota$  are mutual inverses.

The corresponding proofs for positive and negative spectral subspaces are identical.  $\square$

As a corollary of this argument we can show that  $\psi\mathbf{X}_A^\mathbf{p}$  is a retract of  $X^\mathbf{p}$ . This will be crucial in identifying interpolants.

**Corollary 6.1.18.** *Fix  $\mathbf{p}$ ,  $\psi$ , and  $\nu$  as in Proposition 6.1.17, and let  $\iota: \mathbb{X}_A^\mathbf{p} \rightarrow \mathbf{X}$  be a completion of  $\mathbb{X}_A^\mathbf{p}$ . Then the map  $\mathbf{Q}_{\psi,A}^\iota \mathbf{S}_{\psi,A}^\iota: X^\mathbf{p} \rightarrow \psi\mathbf{X}_A^\mathbf{p}$  is the extension of the projection  $\mathbf{Q}_{\psi,A} \mathbf{S}_{\nu,A}: X^2 \cap X^\mathbf{p} \rightarrow \mathbf{Q}_{\psi,A} \mathbb{X}_A^\mathbf{p}$  in the appropriate topology (hence independent of  $\iota$ ), and it is a projection onto  $\psi\mathbf{X}_A^\mathbf{p}$ . The same statements hold for spectral subspaces.*

Therefore we can write  $\mathbf{Q}_{\psi,A} \mathbf{S}_{\nu,A} := \mathbf{Q}_{\psi,A}^\iota \mathbf{S}_{\psi,A}^\iota$  to denote this extension.

Now that we have thought hard enough about completions, we can extend the duality and boundedness results of the previous sections.

**Proposition 6.1.19** (Duality). *Let  $\mathbf{p}$  be a finite exponent, and let  $\psi, \nu \in \Psi_\infty^\infty$  be Calderón siblings. Then the  $X^2$  inner product identifies  $\tilde{\nu}\mathbf{X}_{A*}^{\mathbf{p}'}$  as the Banach space dual of  $\psi\mathbf{X}_A^\mathbf{p}$ , and also identifies  $\tilde{\nu}\mathbf{X}_{A*}^{\mathbf{p}',\pm}$  as the Banach space dual of  $\psi\mathbf{X}_A^{\mathbf{p},\pm}$ .*

*Proof.* If  $f \in \psi\mathbf{X}_A^\mathbf{p}$  and  $g \in \tilde{\nu}\mathbf{X}_{A*}^{\mathbf{p}'}$ , then we immediately have

$$|\langle f, g \rangle_{X^2}| \leq \|f\|_{X^\mathbf{p}} \|g\|_{X^{\mathbf{p}'}} = \|f\|_{\psi\mathbf{X}_A^\mathbf{p}} \|g\|_{\tilde{\nu}\mathbf{X}_{A*}^{\mathbf{p}'}} ,$$

so every  $g \in \tilde{\nu}\mathbf{X}_{A*}^{\mathbf{p}'}$  induces a bounded linear functional on  $\mathbf{X}_A^\mathbf{p}$ .

Conversely, suppose  $\varphi \in (\psi\mathbf{X}_A^\mathbf{p})'$ . Then we can define a bounded linear functional  $\Phi \in (X^\mathbf{p})'$  by

$$\Phi(F) := \varphi(\mathbf{Q}_{\psi,A} \mathbf{S}_{\nu,A} F)$$

for all  $F \in X^\mathbf{p}$ . By  $X$ -space duality, there exists a function  $G_\Phi \in X^{\mathbf{p}'}$  such that

$$\langle F, G_\Phi \rangle_{X^2} = \Phi(F)$$

for all  $F \in X^\mathbf{p}$ , which satisfies

$$\|G_\Phi\|_{X^{\mathbf{p}'}} \simeq \|\Phi\|_{(X^\mathbf{p})'} \lesssim \|\varphi\|_{(\psi\mathbf{X}_A^\mathbf{p})'} .$$

Hence for all  $f \in \psi \mathbf{X}_A^{\mathbf{p}}$  we have

$$\langle f, (\mathbf{Q}_{\psi,A} \mathbf{S}_{\nu,A})^* G_{\Phi} \rangle_{X^2} = \langle f, G_{\Phi} \rangle_{X^2} = \Phi(f) = \varphi(f)$$

since  $f = \mathbf{Q}_{\psi,A} \mathbf{S}_{\nu,A} f$ . Since  $\mathbf{Q}_{\psi,A} \mathbf{S}_{\nu,A}$  is the continuous extension of  $\mathbf{Q}_{\psi,A} \mathbf{S}_{\nu,A}$  from  $X^2 \cap X^{\mathbf{p}}$  to  $X^{\mathbf{p}}$ , and since  $(\mathbf{Q}_{\psi,A} \mathbf{S}_{\nu,A})^* = \mathbf{Q}_{\tilde{\nu},A^*} \mathbf{S}_{\tilde{\psi},A^*}$  on  $X^2$ , we find that  $(\mathbf{Q}_{\psi,A} \mathbf{S}_{\nu,A})^* = \mathbf{Q}_{\tilde{\nu},A^*} \mathbf{S}_{\tilde{\psi},A^*}$ . Therefore we have

$$\varphi(f) = \langle f, G_{\varphi} \rangle_{X^2}$$

for all  $f \in \psi \mathbf{X}_A^{\mathbf{p}}$ , where  $G_{\varphi} = \mathbf{Q}_{\tilde{\nu},A^*} \mathbf{S}_{\tilde{\psi},A^*} G_{\Phi} \in \tilde{\nu} \mathbf{X}_{A^*}^{\mathbf{p}'}$ . Furthermore we have

$$\begin{aligned} \|G_{\varphi}\|_{\tilde{\nu} \mathbf{X}_{A^*}^{\mathbf{p}'}} &= \left\| \mathbf{Q}_{\tilde{\nu},A^*} \mathbf{S}_{\tilde{\psi},A^*} G_{\Phi} \right\|_{X^{\mathbf{p}'}} \\ &\lesssim \|\varphi\|_{(\psi \mathbf{X}_A^{\mathbf{p}})'} . \end{aligned}$$

As with every other result in this section, the same proof works for spectral subspaces.  $\square$

**Proposition 6.1.20** (Boundedness of functional calculus). *Let  $\mathbf{p}$  be an exponent,  $\delta \in \mathbb{R}$ , and  $\eta \in \Psi_{-\delta}^{\delta}$ . Suppose  $\iota_1: \mathbb{X}_A^{\mathbf{p}} \rightarrow \mathbf{X}$  and  $\iota_2: \mathbb{X}_A^{\mathbf{p}+\delta} \rightarrow \mathbf{Y}$  are completions. Then  $\eta(A)$  extends to a bounded operator  $\widetilde{\eta(A)}: \mathbf{X} \rightarrow \mathbf{Y}$ , in the sense that the diagram*

$$\begin{array}{ccc} \mathcal{D}(\eta(A)) \cap \mathbb{X}_A^{\mathbf{p}} & \xrightarrow{\eta(A)} & \mathbb{X}_A^{\mathbf{p}+\delta} \\ \iota_1 \downarrow & & \downarrow \iota_2 \\ \mathbf{X} & \xrightarrow{\widetilde{\eta(A)}} & \mathbf{Y} \end{array}$$

*commutes, and that*

$$\left\| \widetilde{\eta(A)} f \right\|_{\mathbf{Y}} \lesssim \|\eta\|_{\Psi_{-\delta}^{\delta}} \|f\|_{\mathbf{X}} . \quad (6.7)$$

*for all  $f \in \mathbf{X}$ . Similar results hold for spectral subspaces.*

*Proof.* Since  $\mathcal{D}(\eta(A))$  is dense in  $\mathbb{X}_A^2 = \overline{\mathcal{R}(A)}$  and since  $\mathbb{X}_A^2 \cap \mathbb{X}_A^{\mathbf{p}}$  is dense in  $\mathbb{X}_A^{\mathbf{p}}$  (Corollary 6.1.7 for finite exponents, duality for infinite exponents using the weak-star topology), we have that  $\mathcal{D}(\eta(A)) \cap \mathbb{X}_A^{\mathbf{p}}$  is dense in  $\mathbb{X}_A^{\mathbf{p}}$ . The result then follows from Theorem 6.1.11 and the universal property of completions.  $\square$

*Remark 6.1.21.* Evidently ‘completed’ versions of Corollaries 6.1.12, 6.1.13, and 6.1.14 can be formulated.

*Remark 6.1.22.* In the situation of Proposition 6.1.20 we will use the symbol  $\eta(A)$  to denote both the original operator  $\mathcal{D}(\eta(A)) \cap \mathbb{X}_A^{\mathbf{p}} \rightarrow \mathbb{X}_A^{\mathbf{p}+\delta}$  and its extension to completions  $\mathbf{X} \rightarrow \mathbf{Y}$ . This will not cause any ambiguity, but one should be careful.

Finally we can state the interpolation theorem for canonical completions of pre-Hardy–Sobolev and pre-Besov spaces. Having established so much abstract theory, this is now a simple consequence of the interpolation results for tent spaces and  $Z$ -spaces.

**Theorem 6.1.23** (Interpolation of completions). *Fix  $0 < \theta < 1$  and  $\psi \in \Psi_\infty^\infty$ . Let  $\mathbf{p}$  and  $\mathbf{q}$  be exponents.*

(i) *Suppose  $j(\mathbf{p}), j(\mathbf{q}) \geq 0$ , with equality for at most one exponent. Then we have the identification*

$$[\psi \mathbf{H}_A^{\mathbf{p}}, \psi \mathbf{H}_A^{\mathbf{q}}]_\theta = \psi \mathbf{H}_A^{[\mathbf{p}, \mathbf{q}]_\theta}.$$

(ii) *Suppose  $i(\mathbf{p}), i(\mathbf{q}) \geq 1$ , with  $\mathbf{p}$  and  $\mathbf{q}$  not both infinite. Then we have the identification*

$$[\psi \mathbf{B}_A^{\mathbf{p}}, \psi \mathbf{B}_A^{\mathbf{q}}]_\theta = \psi \mathbf{B}_A^{[\mathbf{p}, \mathbf{q}]_\theta}.$$

(iii) *Suppose  $\theta(\mathbf{p}) \neq \theta(\mathbf{q})$ . Then we have the identification*

$$(\psi \mathbf{X}_A^{\mathbf{p}}, \psi \mathbf{X}_A^{\mathbf{q}})_{\theta, p_\theta} = \psi \mathbf{B}_A^{[\mathbf{p}, \mathbf{q}]_\theta}$$

where  $p_\theta = i([\mathbf{p}, \mathbf{q}]_\theta)$ .

*Proof.* Fix a Calderón sibling  $\nu \in \Psi_\infty^\infty$  of  $\psi$ . By Corollary 6.1.18 the map  $\mathbb{Q}_{\psi, A} \mathbf{S}_{\nu, A}$  extends to a map  $\mathbf{Q}_{\psi, A} \mathbf{S}_{\nu, A}: X^{\mathbf{p}} + X^{\mathbf{q}} \rightarrow \psi \mathbf{X}_A^{\mathbf{p}} + \psi \mathbf{X}_A^{\mathbf{q}}$  which restricts to projections  $X^{\mathbf{p}} \rightarrow \psi \mathbf{X}_A^{\mathbf{p}}$  and  $X^{\mathbf{q}} \rightarrow \psi \mathbf{X}_A^{\mathbf{q}}$ . Therefore by the retraction/coretraction interpolation theorem (see [89, §1.2.4]),<sup>3</sup> for all interpolation functors  $\mathcal{F}$  we have

$$\mathcal{F}(\mathbf{X}_A^{\mathbf{p}}, \mathbf{X}_A^{\mathbf{q}}) = \mathbf{Q}_{\psi, A} \mathbf{S}_{\nu, A} \mathcal{F}(X^{\mathbf{p}}, X^{\mathbf{q}}).$$

The results then follow from Corollary 6.1.18 and the interpolation theorems 5.1.12, 5.1.30, and 5.1.31.  $\square$

#### 6.1.4 The Cauchy operator on general adapted spaces

Recall the function  $\text{sgp} = [z \mapsto e^{-[z]}]$ . This function is in  $\Psi_0^\infty \subset H^\infty$ , and therefore for all  $t > 0$  the operator  $e^{-t[A]}$  is defined and bounded on  $\overline{\mathcal{R}(A)}$ . By Corollary 6.1.12, for all exponents  $\mathbf{p}$  we have that  $e^{-t[A]}$  is bounded on  $\mathbb{X}_A^{\mathbf{p}}$ . Furthermore,

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<sup>3</sup>This is only stated for Banach spaces in the given reference. The only property specific to Banach spaces which is needed is the validity of the closed graph theorem, which also holds for quasi-Banach spaces [56, §2], so the proof goes through even for quasi-Banach spaces.

when restricted to the positive or negative spectral subspace, the operator  $\mathbb{Q}_{\text{sgp},A}$  coincides with the Cauchy operator  $C_A^\pm$ , which produces solutions to the Cauchy problem associated with  $A$  on  $\mathbb{R}_+$  or  $\mathbb{R}_-$  respectively.

Given a completion  $\mathbf{X}$  of  $\mathbb{X}_A^{\mathbf{p}}$ , each of the operators  $e^{-t[A]}: \mathbb{X}_A^{\mathbf{p}} \rightarrow \mathbb{X}_A^{\mathbf{p}}$  and the spectral projections  $\chi^\pm(A)$  extend to maps  $\mathbf{X} \rightarrow \mathbf{X}$  (in the sense of Proposition 6.1.20), and by means of these maps we can extend the Cauchy operators  $C_A^\pm$  to maps

$$\mathbf{C}_A^\pm: \mathbf{X} \rightarrow L^\infty(\mathbb{R}_\pm : \mathbf{X}^\pm).$$

Note that we construct these operators by extending each operator  $e^{-t[A]}\chi^\pm(A)$  by boundedness, rather than by extending the Cauchy operators directly. Similarly we can define

$$\mathbf{Q}_{\text{sgp},A}: \mathbf{X} \rightarrow L^\infty(\mathbb{R}_+ : \mathbf{X}).$$

**Proposition 6.1.24** (Properties of Cauchy extensions). *Let  $\mathbf{p}$  be an exponent, and fix a completion  $\mathbf{X}$  of  $\mathbb{X}_A^{\mathbf{p}}$ .<sup>4</sup> Then for all  $f \in \mathbf{X}$  the extension  $\mathbf{Q}_{\text{sgp},A}f$  is in  $C^\infty(\mathbb{R}_+ : \mathbf{X})$ , and if  $f \in \chi^\pm(A)\mathbf{X}$ , then the Cauchy extension  $\mathbf{C}_A^\pm f$  solves the Cauchy equation*

$$\partial_t \mathbf{C}_A^\pm f \pm A \mathbf{C}_A^\pm f = 0$$

*strongly in  $C^\infty(\mathbb{R}_\pm : \mathbf{X})$ . Furthermore for all  $f \in \mathbf{X}$  we also the limits*

$$\lim_{t \rightarrow 0} \mathbf{Q}_{\text{sgp},A}f(t) = f \quad \text{and} \quad \lim_{t \rightarrow \infty} \mathbf{Q}_{\text{sgp},A}f(t) = 0. \quad (6.8)$$

*Proof.* First we will prove the limit results. These reduce to the case of finite exponents  $\mathbf{p}$ , as for infinite  $\mathbf{p}$  we can deduce the limits (6.8) by testing against  $\mathbb{X}_{A*}^{\mathbf{p}'}$ . Furthermore, by density, it suffices to prove the limits

$$\lim_{t \rightarrow 0} e^{-t[A]}f = f \quad \text{and} \quad \lim_{t \rightarrow \infty} e^{-t[A]}f = 0$$

for  $f \in \mathbb{X}_A^{\mathbf{p}}$ .

For  $f \in \mathbb{H}_A^{\mathbf{p}}$ , these follow from arguments almost identical to those of [16, Propositions 4.5 and 4.6], the only difference being the presence of the weight  $\kappa^{-\theta(\mathbf{p})}$ , which does not change the argument. Now fix an exponent  $\mathbf{q} \neq \mathbf{p}$  such that  $\mathbf{q} \hookrightarrow \mathbf{p}$ , so that  $\mathbb{H}_A^{\mathbf{q}} \hookrightarrow \mathbb{B}_A^{\mathbf{p}}$  (Proposition 6.1.9). For  $f \in \mathbb{H}_A^{\mathbf{q}}$ , we then have

$$\lim_{t \rightarrow 0} \|e^{-t[A]}f - f\|_{\mathbb{B}_A^{\mathbf{p}}} \lesssim \lim_{t \rightarrow 0} \|e^{-t[A]}f - f\|_{\mathbb{H}_A^{\mathbf{q}}} = 0$$

---

<sup>4</sup>Recall that we mean a weak-star completion when  $\mathbf{p}$  is infinite, and in this case we use the weak-star topology on  $\mathbf{X}$ .

and

$$\lim_{t \rightarrow \infty} \|e^{-t[A]} f\|_{\mathbb{B}_A^{\mathbf{p}}} \lesssim \lim_{t \rightarrow \infty} \|e^{-t[A]} f\|_{\mathbb{H}_A^{\mathbf{q}}} = 0.$$

Since  $\mathbb{H}_A^{\mathbf{q}}$  is dense in  $\mathbb{B}_A^{\mathbf{p}}$  (Corollary 6.1.7), these limits hold for all  $f \in \mathbb{B}_A^{\mathbf{p}}$ .

Now we will prove the smoothness result. It suffices to work with  $f \in \chi^+(A)\mathbf{X}$  here, as the result for  $\chi^-(A)\mathbf{X}$  uses the same argument, and the general result follows from the decomposition  $\mathbf{X} = \chi^+(A)\mathbf{X} \oplus \chi^-(A)\mathbf{X}$ . First observe that the function  $\Phi: \mathbb{R}_+ \rightarrow H^\infty$  defined by  $\Phi(t) = [z \mapsto e^{-t[z]}]$  is smooth with Fréchet derivative  $D_t\Phi: \mathbb{R} \rightarrow H^\infty$  given by  $D_t\Phi(\tau) = [z \mapsto -\tau[z]e^{-t[z]}]$ . Next, note that the map  $\Omega_A: H^\infty \rightarrow \mathcal{L}(\chi^+(A)\mathbf{X})$  with  $\Omega_A(f) = f(A)$  is linear and bounded in the strong topology (Proposition 6.1.20). By the chain rule, the composition of these maps is smooth, with Fréchet derivative

$$D_t(\Omega_A \circ \Phi)(\tau) = \Omega_A \circ D_t\Phi(\tau) = -\tau A e^{-tA}.$$

We can then write

$$\partial_t \mathbf{C}_A^+ f(t) = D_t(\Omega_A \circ \Phi)(1)f = -A e^{-tA} f = -A \mathbf{C}_A^+ f(t),$$

which completes the proof.  $\square$

Now we must address the question of whether or not  $C_A$  maps  $\mathbb{X}_A^{\mathbf{p}}$  into  $X^{\mathbf{p}}$ . This would imply that one can construct  $C^\infty(\mathbb{R}_\pm : \mathbb{X}_A^{\mathbf{p}})$  solutions<sup>5</sup> to  $(\text{CR})_A$  which are in  $X^{\mathbf{p}}$  with given initial data in  $\mathbb{X}_A^{\mathbf{p}, \pm}$ . It turns out that this is only reasonable when  $\theta(\mathbf{p}) < 0$ . For  $i(\mathbf{p}) \leq 2$  we already know everything we need to prove this; for  $i(\mathbf{p}) > 2$  we need more information (see Subsection 6.2.2).

**Theorem 6.1.25** (Cauchy characterisation of adapted spaces,  $i(\mathbf{p}) \leq 2$ ). *Let  $\mathbf{p}$  be an exponent with  $i(\mathbf{p}) \leq 2$  and  $\theta(\mathbf{p}) < 0$ . Then for all  $f \in \overline{\mathcal{R}(A)}$ ,*

$$\|f\|_{\mathbb{X}_A^{\mathbf{p}}} \simeq \|\mathbb{Q}_{\text{sgp}, A} f\|_{X^{\mathbf{p}}}.$$

*Proof.* By Proposition 6.1.3, we have

$$\text{sgp} \in \Psi_0^\infty \subset \Psi_{\theta(\mathbf{p})+}^{(-\theta(\mathbf{p})+n|\frac{1}{2}-j(\mathbf{p})|)+} \cap H^\infty \subset \Psi(\mathbb{X}_A^{\mathbf{p}}),$$

which yields the result.  $\square$

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<sup>5</sup>This solution concept does not always agree with the  $L_{\text{loc}}^2$  solution concept that we are really interested in. This is discussed further in Subsection 7.3.1.

*Remark 6.1.26.* The estimate

$$\|f\|_{\mathbb{X}_A^{\mathbf{p}}} \lesssim \|\mathbb{Q}_{\text{sgp}, A} f\|_{X^{\mathbf{p}}}$$

holds for all  $\mathbf{p}$ , as can be shown by a Calderón reproducing argument as in the proof of Proposition 6.1.3. The reverse estimate need not hold in general.

We will need the following technical lemmas in Section 7.3.

**Lemma 6.1.27.** *For every  $M > 0$ , there exist functions  $\varphi^+, \varphi^- \in H^\infty$  such that  $(\varphi_s^\pm(A))_{s>0}$  satisfies off-diagonal estimates of order  $M$ ,  $\varphi_s^\pm(A) = e^{-s[A]}$  on the corresponding spectral subspace  $\mathbb{X}_A^{2,\pm}$ , and  $\lim_{s \rightarrow 0} \varphi_s^\pm(A) = I$  in the  $L^2$ -strong operator topology.*

For a proof, see [15, Lemma 15.1], noting that  $\mathbb{H}_A^{2,\pm} = \mathbb{B}_A^{2,\pm}$ . Although this result is stated for  $A \in \{DB, BD\}$  there, the proof only uses the standard assumptions.

**Corollary 6.1.28.** *Let  $p \in (0, \infty]$ . Suppose  $f \in \mathbb{X}_A^\pm \cap E^p$ . Then  $C_A^\pm f(t) \in E^p$  for each  $t \in \mathbb{R}_\pm$ , and if  $p < \infty$  then  $\lim_{t \rightarrow 0} C_A^\pm f(t) = f$  in  $E^p$ .*

*Proof.* Choose functions  $\varphi^\pm$  as in Lemma 6.1.27, such that  $(\varphi_s^\pm(A))_{s>0}$  satisfies off-diagonal estimates of large order. By Proposition 5.2.9, the operators  $\varphi_t^\pm(A)$  are bounded on  $E^p$ . Since  $\varphi_t^\pm(A) = e^{-t[A]}$  on  $\mathbb{X}_A^{2,\pm}$ , we have  $C_A^\pm f(t) = e^{-t[A]} f \in E^p$  for all  $t \in \mathbb{R}_\pm$ . The limit statement follows from Lemma 6.1.27 (which gives strong convergence in  $L^2$ ) and Proposition 5.2.10 (which improves this to  $E^p$ ).  $\square$

## 6.2 Spaces adapted to perturbed Dirac operators

We now begin to work with  $\mathbb{C}^{m(1+n)}$ -valued functions for some fixed  $m \in \mathbb{N}$ . When applying results of the previous sections, we implicitly take  $N = m(1+n)$ .

In this section, we fix the Dirac operator  $D$  and consider multipliers  $B$  as in Subsection 4.1.2 of the introduction. Recall that the perturbed Dirac operators  $DB$  and  $BD$  then satisfy the Standard Assumptions (Theorem 5.2.12). Furthermore,  $\mathcal{R}(DB) = \mathcal{R}(D)$ , and the restrictions  $DB|_{\overline{\mathcal{R}(DB)}}$  and  $BD|_{\overline{\mathcal{R}(BD)}}$  are similar under conjugation by  $B|_{\overline{\mathcal{R}(DB)}}$  [17, Proposition 2.1]. Consequently, whenever  $f \in \mathcal{D}(D) \cap \overline{\mathcal{R}(BD)}$  and  $\varphi \in H^\infty$  we have

$$D\varphi(BD)f = \varphi(DB)Df.$$

We will refer to this principle as *similarity of functional calculi* and use it repeatedly.

### 6.2.1 Identification of spaces adapted to $D$ , $DB$ , and $BD$

The operators  $DB$  and  $BD$  satisfy the Standard Assumptions, so we can define pre-Besov–Hardy–Sobolev spaces  $\mathbb{X}_{DB}^{\mathbf{p}}$ . The case  $B = I$  yields  $\mathbb{X}_D^{\mathbf{p}}$ .

For a certain range of exponents  $\mathbf{p}$  that we denote by  $I_{\max}$ , the spaces  $\mathbb{X}_D^{\mathbf{p}}$  may be identified as projections of classical smoothness spaces (to be shown in Proposition 6.2.1). Recall that we use the notation  $\mathbf{X}^{\mathbf{p}}$  to denote classical smoothness spaces, as in Subsection 5.1.5. These spaces are all contained in  $\mathcal{Z}'(\mathbb{R}^n)$ , the space of tempered distributions modulo polynomials.

In [16, Theorem 4.16] it is shown that for  $p \in (n/(n+1), \infty)$  we have an identification

$$\mathbb{H}_D^{(p,0)} \simeq \mathbb{P}_D(\mathbf{H}^{(p,0)} \cap L^2) \subset \mathcal{Z}'(\mathbb{R}^n)$$

where  $\mathbb{P}_D$  is the bounded projection from  $L^2(\mathbb{R}^n)$  onto  $\overline{\mathcal{R}(D)}$ . Since  $\mathbb{P}_D$  extends boundedly to the spaces  $\mathbf{H}^{(p,0)}$  by virtue of being a Fourier multiplier within the scope of the Mikhlin multiplier theorem (see [90, Theorem 5.2.2] and [53, Proposition 4.4]), we may write

$$\mathbf{H}_D^{(p,0)} = \mathbb{P}_D(\mathbf{H}^{(p,0)}) = \mathbf{H}^{(p,0)} \cap D\mathcal{Z}' \subset \mathcal{Z}'(\mathbb{R}^n),$$

thus providing a completion of  $\mathbb{H}_D^{(p,0)}$  within the space of distributions modulo polynomials. Hence if we have an identification  $\mathbb{H}_{DB}^{(p,0)} \simeq \mathbb{H}_D^{(p,0)}$ , we can find a completion of  $\mathbb{H}_{DB}^{(p,0)}$  in  $\mathcal{Z}'$ .

Furthermore, combining [16, Lemma 11.6] with Corollary 6.1.14 shows that for all  $p \in (1, \infty)$  we have

$$\mathbb{H}_D^{(p,-1)} \simeq \mathbb{P}_D(\mathbf{H}^{(p,-1)} \cap L^2) \cap D\mathcal{Z}' \subset \mathcal{Z}'(\mathbb{R}^n),$$

and for all  $\alpha \in [0, 1)$  we have

$$\mathbb{H}_D^{(\infty,0;\alpha)} \simeq \mathbb{P}_D(\mathbf{H}^{(\infty,0;\alpha)} \cap L^2) \subset \mathcal{Z}'(\mathbb{R}^n),$$

so by the same argument we may write

$$\mathbf{H}_D^{(p,-1)} \simeq \mathbb{P}_D(\mathbf{H}^{(p,-1)}) = \mathbf{H}^{(p,-1)} \cap D\mathcal{Z}' \subset \mathcal{Z}'(\mathbb{R}^n)$$

and

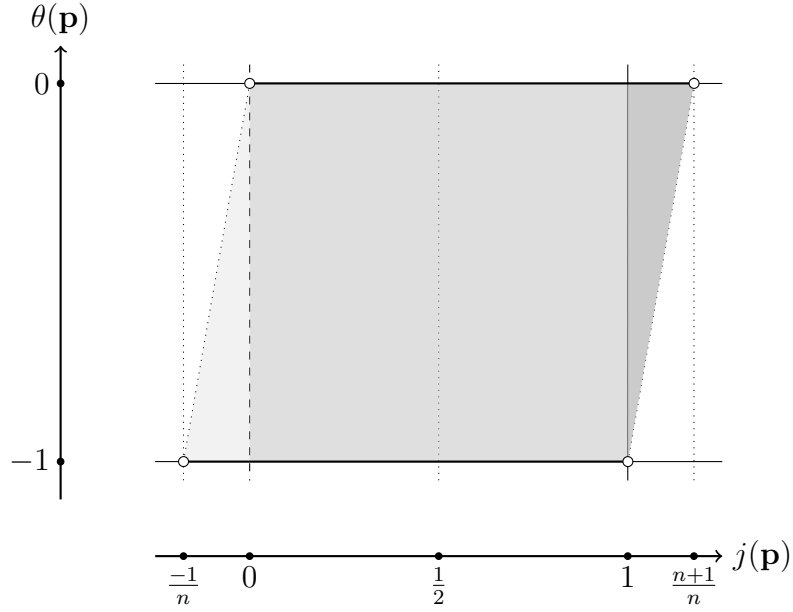
$$\mathbf{H}_D^{(\infty,0;\alpha)} \simeq \mathbb{P}_D(\mathbf{H}^{(\infty,0;\alpha)}) = \mathbf{H}^{(\infty,0;\alpha)} \cap D\mathcal{Z}' \subset \mathcal{Z}'(\mathbb{R}^n)$$

(where  $\mathbb{P}_D$  is extended by duality).

We can interpolate between these observations to yield an identification of the spaces  $\mathbf{H}_D^{\mathbf{p}}$  in a restricted (but for our applications, sufficiently large) range of  $\mathbf{p}$ .



Figure 6.1: The region  $I_{\max}$  on which  $\mathbf{H}_D^{\mathbf{p}} \simeq \mathbf{H}^{\mathbf{p}} \cap D\mathcal{Z}'$ .



**Theorem 6.2.1** (Identification of  $D$ -adapted spaces). *Suppose  $\mathbf{p}$  is in the region  $I_{\max}$  pictured in Figure 6.1. Then*

$$\mathbb{H}_D^{\mathbf{p}} \simeq \mathbb{P}_D(\mathbf{H}^{\mathbf{p}} \cap L^2).$$

*Furthermore, if  $\mathbf{p}$  is in the interior of  $I_{\max}$ , then*

$$\mathbb{B}_D^{\mathbf{p}} \simeq \mathbb{P}_D(\mathbf{B}^{\mathbf{p}} \cap L^2).$$

We abuse notation by writing  $\mathbf{X}_D^{\mathbf{p}} = \mathbb{P}_D(\mathbf{X}^{\mathbf{p}})$ .

To prove this we will need the following lemma.

**Lemma 6.2.2.** *Suppose that  $f \in \mathcal{R}(D)$  (note that we do not take the closure of the range here). Then there exists  $g \in \mathcal{D}(D) \cap \overline{\mathcal{R}(D)}$  such that  $f = Dg$  and*

$$\|f\|_{\mathbf{X}^{\mathbf{p}}} \simeq \|g\|_{\mathbf{X}^{\mathbf{p}+1}}.$$

*for all exponents  $\mathbf{p}$ .*

*Proof.* Since  $f \in \mathcal{R}(D)$  there exists  $\tilde{g} \in \mathcal{D}$  such that  $f = D\tilde{g}$ . Let  $g = \mathbb{P}_D\tilde{g}$ . Since the projection  $\mathbb{P}_D$  is along  $\mathcal{N}(D)$ , we have  $f = Dg$  also. The estimate

$$\|f\|_{\mathbf{X}^{\mathbf{p}}} \lesssim \|g\|_{\mathbf{X}^{\mathbf{p}+1}}$$

then follows since  $D$  is a first-order homogeneous differential operator. To obtain the reverse estimate, we need to invert  $D$  on  $\overline{\mathcal{R}(D)}$ . Let

$$T = D\Delta^{-1}\mathbb{P}_D + (-\Delta)^{-1/2}(I - \mathbb{P}_D).$$

One can show that  $T$  is a homogeneous Fourier multiplier of order  $-1$ , and hence maps  $\mathbf{X}^{\mathbf{p}}$  to  $\mathbf{X}^{\mathbf{p}+1}$ . Furthermore,  $T\mathbb{P}_D$  inverts  $D$  on  $\mathcal{R}(D)$ , and so we have the estimate

$$\|g\|_{\mathbf{X}^{\mathbf{p}+1}} \lesssim \|f\|_{\mathbf{X}^{\mathbf{p}}}$$

which completes the proof.  $\square$

*Proof of Theorem 6.2.1.* When  $\mathbf{p}$  is finite, this follows directly from the identification of complex and real interpolants of the spaces  $\mathbf{H}^{\mathbf{p}}$  (Theorem 5.1.52).

Now suppose  $\mathbf{p}$  is infinite. Observe that the subregion of  $I_{\max}$  consisting of infinite exponents (the lightest shaded region, including the dashed line) is precisely the  $\heartsuit$ -dual region of the darkest shaded region, including the solid line. Therefore for all infinite  $\mathbf{p} \in I_{\max}$ ,  $\mathbf{p}^\heartsuit$  is finite and in  $I_{\max}$ . If  $f = Dg$  for some  $g \in \mathcal{D}(D) \cap \overline{\mathcal{R}(D)}$  as in Lemma 6.2.2, then we have

$$\|f\|_{\mathbb{X}_D^{\mathbf{p}}} \simeq \|g\|_{\mathbb{X}_D^{\mathbf{p}+1}} \tag{6.9}$$

$$\begin{aligned} &\simeq \sup_{h \in \mathbb{X}_D^{\mathbf{p}^\heartsuit}} |\langle g, h \rangle| \\ &\simeq \sup_{h \in \mathbf{X}^{\mathbf{p}^\heartsuit} \cap L^2} |\langle g, \mathbb{P}_D h \rangle| \\ &= \sup_{h \in \mathbf{X}^{\mathbf{p}^\heartsuit} \cap L^2} |\langle \mathbb{P}_D g, h \rangle| \end{aligned} \tag{6.10}$$

$$\simeq \|g\|_{\mathbf{X}^{\mathbf{p}+1}} \tag{6.11}$$

$$\simeq \|f\|_{\mathbf{X}^{\mathbf{p}}} . \tag{6.12}$$

with all suprema taken over appropriately normalised elements. Line (6.9) follows from Corollary 6.1.14. In (6.10) we use orthogonality of the decomposition  $L^2 = \mathcal{N}(D) \oplus \overline{\mathcal{R}(D)}$ . In line (6.11) we remove the projection by using that  $g \in \overline{\mathcal{R}(D)}$ . In (6.12) we use the conclusion of Lemma 6.2.2, which follows from our choice of  $g$ . Therefore by weak-star density, we get  $\mathbb{X}_D^{\mathbf{p}} \simeq \mathbb{P}_D(\mathbf{X}^{\mathbf{p}} \cap L^2)$  (the projection comes from the fact that  $f \in \mathcal{R}(D)$  in this estimate).  $\square$

Now we shall discuss spaces adapted to  $DB$ , and the range of exponents for which they may be identified with spaces adapted to  $D$ . As shown in the introduction, the following ‘identification region’ plays a central role in the theorems of Chapter 7.

**Definition 6.2.3.** We define

$$I(\mathbf{X}, DB) := \left\{ \mathbf{p} \in I_{\max} \cap \mathbf{E}_{\text{fin}} : \|f\|_{\mathbb{X}_{DB}^{\mathbf{p}}} \simeq \|f\|_{\mathbb{X}_D^{\mathbf{p}}} \text{ for all } f \in \overline{\mathcal{R}(DB)} = \overline{\mathcal{R}(D)} \right\}. \quad (6.13)$$

and for  $s \in \mathbb{R}$ ,

$$I_s(\mathbf{X}, DB) := \{i(\mathbf{p}) : \mathbf{p} \in I(\mathbf{X}, DB) : \theta(\mathbf{p}) = s\} \subset (0, \infty).$$

Note that  $I(\mathbf{X}, DB)$  is defined to be a set of finite exponents. We could include infinite exponents in this definition, but it is technically more convenient to restrict ourselves to finite exponents.<sup>6</sup> It is also defined to be contained in  $I_{\max}$ , so not only do we have  $\mathbb{X}_{DB}^{\mathbf{p}} = \mathbb{X}_D^{\mathbf{p}}$ , but we also have the identification of  $\mathbb{X}_D^{\mathbf{p}}$  as the projection of a classical space.

We recall a key result of Auscher and Stahlhut, which follows from [16, Theorem 5.1 and Remark 5.2].

**Theorem 6.2.4** (Auscher–Stahlhut). *There exists  $\varepsilon = \varepsilon(B) > 0$  such that  $(2n/(n+2) - \varepsilon, 2 + \varepsilon) \subset I_0(\mathbf{H}, DB)$ . Furthermore, if  $n = 1$ , then  $I_0(\mathbf{H}, DB) = (1/2, \infty)$ .*

We will extend this result to allow for more general exponents of order  $\theta(\mathbf{p}) \in [-1, 0]$ , and also to incorporate Besov spaces.

The  $\heartsuit$ -duality operation on exponents provides a link between  $I(\mathbf{X}, DB)$  and  $I(\mathbf{X}, DB^*)$  (Proposition 6.2.7). We need some preliminary results to establish this link. First we state a local coercivity property of  $B$ , which is proven in [16, Lemma 5.14].

**Lemma 6.2.5.** *For any  $u \in L_{loc}^2$  with  $Du \in L_{loc}^2$  and any ball  $B(x, t) \in \mathbb{R}^n$ , we have*

$$\int_{B(x,t)} |Du|^2 \lesssim_{B,n,N} \int_{B(x,2t)} |BDu|^2 + t^{-2} \int_{B(x,2t)} |u|^2.$$

**Proposition 6.2.6** (Intertwining and regularity shift). *Let  $\mathbf{p}$  be an exponent, and suppose  $f \in \mathbb{X}_{BD}^{\mathbf{p}} \cap \mathcal{D}(D)$ . Then  $Df \in \mathbb{X}_{DB}^{\mathbf{p}-1}$  and*

$$\|Df\|_{\mathbb{X}_{DB}^{\mathbf{p}-1}} \simeq \|f\|_{\mathbb{X}_{BD}^{\mathbf{p}}}.$$

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<sup>6</sup>More precisely: in applications, whenever we deal with infinite exponents, we always consider the space in question as the dual space, and the predual exponent will be in  $I(\mathbf{X}, DB^*)$  (see for example Theorem 7.3.2).

*Proof.* We will only prove the result for  $T^{\mathbf{p}}$  with  $\mathbf{p} = (p, s)$  finite; all other cases are proven by the same argument.

Let  $\psi \in \Psi_{\infty}^0$  be nondegenerate and define  $\tilde{\psi} \in \Psi_{\infty}^0$  by  $\tilde{\psi}(z) = z\tilde{\psi}$ . Then  $\tilde{\psi}(DB)$  maps  $\overline{\mathcal{R}(DB)}$  into  $\mathcal{D}((DB)^{-1})$ . Since  $f \in \mathcal{D}(D)$  we have  $Df \in \mathcal{R}(D) = \mathcal{R}(DB)$ . Using similarity of functional calculi, write

$$\begin{aligned} \|Df\|_{\mathbb{H}_{DB}^{\mathbf{p}-1}} &\simeq \|t \mapsto \psi(tDB)Df\|_{T_{s-1}^p} \\ &= \|t \mapsto D\psi(tBD)f\|_{T_{s-1}^p} \\ &= \left\| t \mapsto D(BD)^{-1}\tilde{\psi}(tBD)f \right\|_{T_s^p}. \end{aligned}$$

For all  $t > 0$  we have

$$(BD)^{-1}\tilde{\psi}(tBD)f \in L^2, D(BD)^{-1}\tilde{\psi}(tBD)f = \psi(tDB)Df \in L^2,$$

so we can apply Lemma 6.2.5 for each  $t > 0$  with  $u = (BD)^{-1}\tilde{\psi}(tBD)f$  as follows: for all  $x \in \mathbb{R}^n$ ,

$$\begin{aligned} &\mathcal{A}(t \mapsto t^{-s}D(BD)^{-1}\tilde{\psi}(tBD)f)(x) \\ &= \left( \int_0^\infty t^{-2s} \int_{B(x,t)} |(D(BD)^{-1}\tilde{\psi}(tBD)f)(y)|^2 dy \frac{dt}{t^{n+1}} \right)^{1/2} \\ &\lesssim \left( \int_0^\infty t^{-2s} \left[ \int_{B(x,2t)} |(\tilde{\psi}(tBD)f)(y)|^2 dy \right. \right. \\ &\quad \left. \left. + \int_{B(x,2t)} |(\psi(tBD)f)(y)|^2 dy \right] \frac{dt}{t^{n+1}} \right)^{1/2} \\ &\lesssim \mathcal{A}(\kappa^{-s}\mathbb{Q}_{\tilde{\psi},BD}f)(x) + \mathcal{A}(\kappa^{-s}\mathbb{Q}_{\psi,BD}f)(x). \end{aligned}$$

Therefore

$$\|Df\|_{\mathbb{H}_{DB}^{\mathbf{p}-1}} \lesssim \|\mathbb{Q}_{\tilde{\psi},BD}f\|_{T^{\mathbf{p}}} + \|\mathbb{Q}_{\psi,BD}f\|_{T^{\mathbf{p}}} \simeq \|f\|_{\mathbb{H}_{BD}^{\mathbf{p}}}.$$

To prove the reverse estimate, using that  $f \in \mathcal{D}(D) = \mathcal{D}(BD)$ , write

$$\begin{aligned} \|f\|_{\mathbb{H}_{BD}^{\mathbf{p}}} &= \|(BD)^{-1}BDf\|_{\mathbb{H}_{BD}^{\mathbf{p}}} \\ &\lesssim \|BDf\|_{\mathbb{H}_{BD}^{\mathbf{p}-1}} \\ &\simeq \|t \mapsto \psi(tBD)BDf\|_{T^{\mathbf{p}-1}} \\ &= \|t \mapsto B\psi(tDB)Df\|_{T^{\mathbf{p}-1}} \\ &\lesssim \|\mathbb{Q}_{\psi,DB}Df\|_{T^{\mathbf{p}-1}} \\ &\simeq \|Df\|_{\mathbb{H}_{DB}^{\mathbf{p}-1}} \end{aligned}$$

using (6.1.14), boundedness of the multiplier  $B$ , and similarity of functional calculi.  $\square$

**Proposition 6.2.7** ( $\heartsuit$ -duality of identification regions). *If  $\mathbf{p} \in I(\mathbf{X}, DB)$ , then  $\|f\|_{\mathbb{X}_{DB}^{\mathbf{p}^\heartsuit}} \simeq \|f\|_{\mathbb{X}_D^{\mathbf{p}^\heartsuit}}$  for all  $f \in \overline{\mathcal{R}(D)}$ . In particular, if  $\mathbf{p}^\heartsuit$  is also finite, then  $\mathbf{p}^\heartsuit \in I(\mathbf{X}, DB^*)$ .*

*Proof.* Suppose  $g \in \mathcal{D}(D) \cap \mathbb{X}_{B^*D}^{\mathbf{p}'}$ . Then arguing similarly to the proof of Theorem 6.2.1,

$$\begin{aligned} \|Dg\|_{\mathbb{X}_{DB^*}^{\mathbf{p}^\heartsuit}} &\simeq \sup_{h \in \mathcal{D}(D) \cap \mathbb{X}_{B^*D}^{(\mathbf{p}^\heartsuit)'}} |\langle Dg, h \rangle| \\ &= \sup_{h \in \mathcal{D}(D) \cap \mathbb{X}_{B^*D}^{\mathbf{p}+1}} |\langle g, Dh \rangle| \\ &\simeq \sup_{h \in \mathbb{X}_{DB^*}^{\mathbf{p}}} |\langle g, h \rangle| \end{aligned} \tag{6.14}$$

$$\simeq \sup_{h \in \mathbb{X}_D^{\mathbf{p}}} |\langle g, h \rangle| \tag{6.15}$$

$$\begin{aligned} &\simeq \|g\|_{\mathbb{X}_D^{\mathbf{p}'}} \\ &\simeq \|Dg\|_{\mathbb{X}_D^{\mathbf{p}^\heartsuit}} \end{aligned} \tag{6.16}$$

with all suprema taken over appropriately normalised elements. The equivalence (6.14) uses Proposition 6.2.6, and then (6.15) uses the assumption on  $\mathbf{p}$ . The final equivalence (6.16) uses Corollary 6.1.14. Since  $D(\mathcal{D}(D) \cap \mathbb{X}_{B^*D}^{\mathbf{p}'})$  is dense<sup>7</sup> in  $\mathbb{X}_{DB^*}^{\mathbf{p}^\heartsuit}$  (by density of  $\mathcal{R}(D)$  in  $\mathbb{X}_D^2$  and Corollary 6.1.7), we are done.  $\square$

The following result then follows immediately from Theorem 6.2.4.

**Corollary 6.2.8.** *There exists  $\varepsilon > 0$  such that*

$$I_{-1}(\mathbf{H}, DB) \supset \begin{cases} (1, \infty) & (n = 1) \\ (2 - \varepsilon, \infty) & (n = 2) \\ (2 - \varepsilon, 2n/(n - 2) + \varepsilon) & (n \geq 3) \end{cases}$$

The main application of our discussion of interpolations and completions of adapted spaces, particularly Theorem 6.1.23, is in showing that  $I(\mathbf{X}, DB)$  is closed under interpolation, and also that information on  $I(\mathbf{H}, DB)$  implies information on  $I(\mathbf{B}, DB)$ .

**Proposition 6.2.9** (Convexity of identification regions). *Let  $\theta \in [0, 1]$ .*

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<sup>7</sup>Weak-star dense when  $\mathbf{p}'$  is infinite.

(i) If  $\mathbf{p}, \mathbf{q} \in I(\mathbf{H}, DB)$ , then  $[\mathbf{p}, \mathbf{q}]_\theta$  is in  $I(\mathbf{H}, DB)$ . Furthermore, if  $\theta(\mathbf{p}) \neq \theta(\mathbf{q})$ , then  $[\mathbf{p}, \mathbf{q}]_\theta$  is in  $I(\mathbf{B}, DB)$ .

(ii) If  $\mathbf{p}, \mathbf{q} \in I(\mathbf{B}, DB)$ , then  $[\mathbf{p}, \mathbf{q}]_\theta \in I(\mathbf{B}, DB)$ .

*Proof.* We will only prove the first part, as the proof of the other parts are identical but with real interpolation replacing complex interpolation.

Suppose  $\mathbf{p}, \mathbf{q} \in I(\mathbf{H}, DB)$ . Let  $\psi, \varphi \in \Psi_\infty^\infty$  be Calderón siblings. First note that we have a map

$$\mathbb{Q}_{\psi, DB} \mathbb{X}_{DB}^{\mathbf{p}} + \mathbb{Q}_{\psi, DB} \mathbb{X}_{DB}^{\mathbf{q}} \xrightarrow{\mathbb{S}_{\varphi, DB}} \mathbb{X}_{DB}^{\mathbf{p}} + \mathbb{X}_{DB}^{\mathbf{q}} \xrightarrow{\text{id}} \mathbf{X}_D^{\mathbf{p}} + \mathbf{X}_D^{\mathbf{q}}$$

(here we use Theorem 6.2.1), which restricts appropriately and which extends by boundedness to

$$\mathbf{S}_{\varphi, DB}: \psi \mathbf{X}_{DB}^{\mathbf{p}} + \psi \mathbf{X}_{DB}^{\mathbf{q}} \rightarrow \mathbf{X}_D^{\mathbf{p}} + \mathbf{X}_D^{\mathbf{q}}.$$

By Proposition 6.1.17, the restrictions of  $\mathbf{S}_{\varphi, DB}$  to  $\psi \mathbf{X}_{DB}^{\mathbf{p}}$  and  $\psi \mathbf{X}_{DB}^{\mathbf{q}}$  are isomorphisms, and their inverses both extend  $\mathbb{Q}_{\psi, DB}: \mathbb{X}_{DB}^{(2,0)} \rightarrow \mathbb{Q}_{\psi, DB} \mathbb{X}_{DB}^{(2,0)}$ . Therefore, by complex interpolation (Theorem 6.1.23) we have an isomorphism

$$\mathbf{S}_{\varphi, DB}: \psi \mathbf{X}_{DB}^{[\mathbf{p}, \mathbf{q}]_\theta} \rightarrow \mathbf{X}_D^{[\mathbf{p}, \mathbf{q}]_\theta}$$

which extends  $\mathbb{S}_{\varphi, DB}: \mathbb{Q}_{\psi, DB} \mathbb{X}_{DB}^{(2,0)} \rightarrow \mathbb{X}_{DB}^{(2,0)} = \mathbf{X}_D^{(2,0)}$ . Hence for all  $f \in \overline{\mathcal{R}(DB)} = \mathbb{X}_{DB}^{(2,0)}$  we have

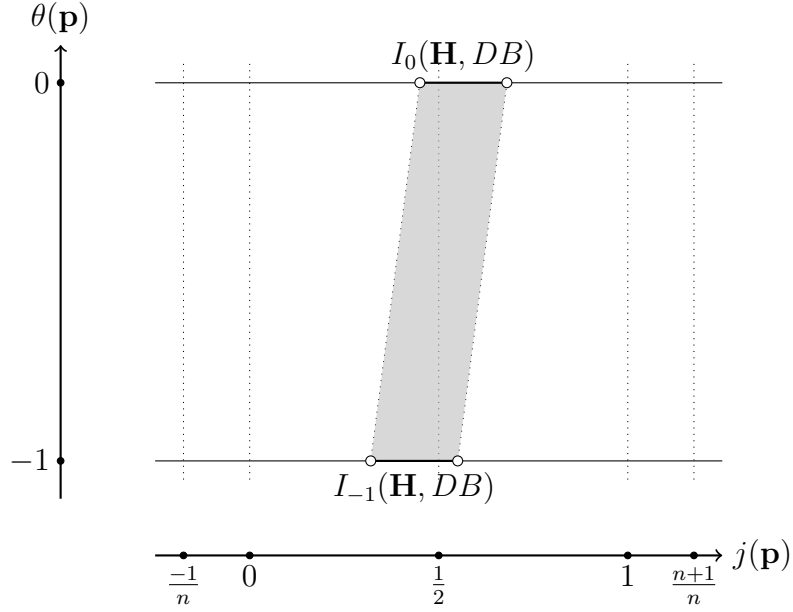
$$\begin{aligned} \|f\|_{\mathbb{X}_{DB}^{[\mathbf{p}, \mathbf{q}]_\theta}} &\simeq \|\mathbb{Q}_{\psi, DB} f\|_{\mathbf{X}_D^{[\mathbf{p}, \mathbf{q}]_\theta}} \\ &\simeq \|\mathbf{S}_{\varphi, DB} \mathbb{Q}_{\psi, DB} f\|_{\mathbf{X}_D^{[\mathbf{p}, \mathbf{q}]_\theta}} \\ &= \|\mathbb{S}_{\varphi, DB} \mathbb{Q}_{\psi, DB} f\|_{\mathbb{X}_D^{[\mathbf{p}, \mathbf{q}]_\theta}} \\ &= \|f\|_{\mathbb{X}_D^{[\mathbf{p}, \mathbf{q}]_\theta}}, \end{aligned}$$

and therefore  $[\mathbf{p}, \mathbf{q}]_\theta \in I(\mathbf{H}, DB)$ .  $\square$

Therefore for every  $B$  we have a region  $I_{\min}$  such that  $I_{\min} \subset I(\mathbf{H}, DB)$  and  $I_{\min}^o \subset I(\mathbf{B}, DB)$ , pictured in Figure 6.2, where lower bounds on  $I_0(\mathbf{H}, DB)$  and  $I_{-1}(\mathbf{H}, DB)$  can be found in Theorem 6.2.4 and Corollary 6.2.8.

*Remark 6.2.10.* If  $\mathbf{p} \in I(\mathbf{X}, DB)$ , then we identify the projected classical Besov–Hardy–Sobolev space  $\mathbb{P}_D(\mathbf{X}^{\mathbf{p}}) = \mathbf{X}^{\mathbf{p}} \cap D\mathcal{Z}'$  as a completion of  $\mathbb{X}_{DB}^{\mathbf{p}}$  via the extension of the identity map  $\mathbb{X}_{DB}^{\mathbf{p}} \rightarrow \mathbb{X}_D^{\mathbf{p}}$ . If  $\mathbf{p}$  is infinite and  $\mathbf{p}^\heartsuit \in I(\mathbf{X}, DB^*)$ , then by Proposition 6.2.7 we may identify  $\mathbb{P}_D(\mathbf{X}^{\mathbf{p}})$  as a weak-star completion of  $\mathbb{X}_{DB}^{\mathbf{p}}$ . We abuse notation by writing  $\mathbf{X}_{DB}^{\mathbf{p}}$  for these completions.

Figure 6.2: The region  $I_{\min} \subset I(\mathbf{H}, DB)$ .



Having made these identifications, note that we do not have equality of  $\mathbf{X}_{DB}^{\mathbf{p},+}$  and  $\mathbf{X}_D^{\mathbf{p},+}$ . The first of these spaces is defined via the spectral projection  $\chi^+(DB)$ , while the second is defined via  $\chi^+(D)$ . However, we do of course have  $\mathbf{X}_{DB}^{\mathbf{p},+} \subset \mathbf{X}_D^{\mathbf{p},+}$ . This will be important in applications to boundary value problems.

*Remark 6.2.11.* For a coefficient matrix  $A$  as in the introduction, if  $B = \hat{A}$ , then  $\hat{A}^* = \tilde{B} := NB^*N$ , where

$$N := \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.$$

Since  $DN = -ND$  and  $N$  acts on  $\overline{\mathcal{R}(D)}$ , the operators  $DB^*$  and  $-D\tilde{B}$  are similar on  $\overline{\mathcal{R}(D)} = \overline{\mathcal{R}(DB^*)} = \overline{\mathcal{R}(D\tilde{B})}$ . Thus all functional calculus properties of  $DB^*$  can be transferred to  $D\tilde{B}$ , and vice versa. This gives natural isomorphisms between  $\mathbb{X}_{DB^*}^{\mathbf{p}}$  and  $\mathbb{X}_{D\tilde{B}}^{\mathbf{p}}$ , and in particular we have  $I(\mathbf{X}, DB^*) = I(\mathbf{X}, D\tilde{B})$ . For further details see [16, §12.2].

### 6.2.2 The Cauchy operator on $DB$ -adapted spaces

This section is devoted to the proof of the following theorem. Recall that the region  $I_{\max}$  is introduced in Theorem 6.2.1.

**Theorem 6.2.12** (Cauchy characterisation of adapted spaces,  $i(\mathbf{p}) > 2$ ). *Let  $\mathbf{p}$  be such that  $i(\mathbf{p}) > 2$ ,  $\theta(\mathbf{p}) \in (-1, 0)$  and  $\mathbf{p}^\heartsuit \in I(\mathbf{X}, DB^*)$ . Then for all  $f \in \overline{\mathcal{R}(DB)}$ ,*

$$\|C_{DB}^+ f\|_{X^{\mathbf{p}}} \lesssim \|f\|_{X^{\mathbf{p}}_D}.$$

*Remark 6.2.13.* The condition  $\mathbf{p}^\heartsuit \in I(\mathbf{X}, DB^*)$  is equivalent to  $\mathbf{p} \in I(\mathbf{X}, DB)$  when  $\mathbf{p}$  is finite (Proposition 6.2.7).

*Remark 6.2.14.* The reverse estimate

$$\|f\|_{X^{\mathbf{p}}_A} \lesssim \|C_A^+ f\|_{X^{\mathbf{p}}}$$

holds for general operators  $A$  (satisfying the standard assumptions) and for all  $\mathbf{p}$  (see Remark 6.1.26). However, we do not know whether Theorem 6.2.12 holds with  $DB$  replaced by  $A$  and without the assumption on  $\mathbf{p}^\heartsuit$ .

In contrast with Theorem 6.1.25, the proof of this theorem is quite long. We thank Pascal Auscher for suggesting this argument.

Before proving the theorem, we establish a technical lemma.

**Lemma 6.2.15.** *Suppose  $\theta \in (-1, 0)$ ,  $g \in \mathcal{D}(D)$ , and  $f = Dg$ . Then for all  $\xi \in \mathbb{R}^n$ ,  $\tau > 0$ , and  $M \in \mathbb{N}$ , we have*

$$\begin{aligned} & \iint_{T(B(\xi, \tau))} |t^{-\theta} (I + itDB)^{-2} f(x)|^2 \frac{dx dt}{t} \\ & \lesssim_M \int_{B(\xi, 4\tau)} \left( \int_{B(\xi, 4\tau)} + \sum_{j=2}^{\infty} 2^{-2j(M - \frac{n}{2} - (1+\theta))} \int_{A(\xi, 2^{j-1}\tau, 2^{j+2}\tau)} \right) G(x, y) dx dy \end{aligned}$$

where

$$G(x, y) := \frac{|g(x) - g(y)|^2}{|x - y|^{n+2(1+\theta)}}.$$

*Proof.* Fix  $\chi_1, \chi \in C_c^\infty(\mathbb{R}^n)$  such that

$$\begin{aligned} \text{supp } \chi_1 &\subset B(\xi, 4\tau), & \chi_1|_{B(\xi, 2\tau)} &\equiv \text{const}, \\ \text{supp } \chi &\subset A(\xi, \tau/2, 4\tau) & \chi|_{A(\xi, \tau, 2\tau)} &\equiv \text{const} \end{aligned}$$

For all  $j \geq 2$  define  $\chi_j(x) := \chi(2^{-j}x)$ , so that  $\text{supp } \chi_j \subset A(\xi, 2^{j-1}\tau, 2^{j+2}\tau)$  and  $\chi_j|_{A(\xi, 2^j\tau, 2^{j+1}\tau)} \equiv 1$ . We can choose the functions  $\chi_1$  and  $\chi$  such that  $\sum_{j=1}^{\infty} \chi_j \equiv 1$ . Let

$$c := \int_{B(\xi, \tau/2)} g.$$



Then we have

$$f = D(g - c) = \sum_{j=1}^{\infty} D(g - c)\chi_j.$$

First we will prove that

$$\begin{aligned} & \iint_{T(B(\xi, \tau))} |t^{-\theta}(I + itDB)^{-2}D(g - c)\chi_1(x)|^2 \frac{dx dt}{t} \\ & \lesssim \int_{B(\xi, 4\tau)} \int_{B(\xi, 4\tau)} G(x, y) dx dy. \end{aligned} \quad (6.17)$$

Since  $\Psi_0^2 \in \Psi_{\theta+}^{-\theta} \cap H^\infty \subset \Psi(\mathbb{X}_{DB}^{(2, \theta)})$  and since  $(2, \theta) \in I(\mathbf{H}, DB)$ , we have

$$\begin{aligned} & \iint_{T(B(\xi, \tau))} |t^{-\theta}(I + itDB)^{-2}D(g - c)\chi_1(x)|^2 \frac{dx dt}{t} \\ & \leq \iint_{\mathbb{R}_+^{1+n}} |t^{-\theta}(I + itDB)^{-2}D(g - c)\chi_1(x)|^2 \frac{dx dt}{t} \\ & \simeq \|D(g - c)\chi_1\|_{\mathbb{H}_{DB}^{(2, \theta)}}^2 \\ & \simeq \|(g - c)\chi_1\|_{\dot{H}_{\theta+1}^2}^2 \\ & \simeq \int_{\mathbb{R}^n} |\mathcal{D}_{\theta+1}^2(g - c)\chi_1(x)|^2 dx \end{aligned}$$

using Lemma 5.1.49 (which is valid since  $2 > 2n/(n + 1 + \theta)$ ) in the last line.

We claim that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|(g - c)\chi_1(z) - (g - c)\chi_1(y)|^2}{|z - y|^{n+2(\theta+1)}} dz dy \lesssim \iint_{B(\xi, 4\tau)^2} \frac{|g(z) - g(y)|^2}{|z - y|^{n+2(\theta+1)}} dz dy, \quad (6.18)$$

from which estimate (6.17) will follow. First observe that if  $y \in B(\xi, \tau)$  and  $z \in B(\xi, \tau/2)$ , then  $\chi_1(z) = \chi_1(y) = 1$  and the estimate (6.18) (restricted to such

$y$  and  $z$ ) follows immediately. Next, we can estimate

$$\begin{aligned}
& \int_{B(\xi, \tau)^c} \int_{B(\xi, \tau/2)} \frac{|(g-c)\chi_1(z) - (g-c)\chi_1(x)|^2}{|z-x|^{n+2(\theta+1)}} dz dx \\
& \lesssim \int_{A(\xi, \tau, 4\tau)} \int_{B(\xi, \tau/2)} \frac{|(g-c)(z)(1-\chi_1(x))|^2}{|z-x|^{n+2(\theta+1)}} dz dx + A \\
& \lesssim_\chi \int_{A(\xi, \tau, 4\tau)} \int_{B(\xi, \tau/2)} |z-x|^{-n-2(\theta+1)} \left( \int_{B(\xi, \tau/2)} |g(z) - g(y)| dy \right)^2 dz dx + A \\
& \lesssim r^n \int_{B(\xi, \tau/2)} |z-y|^{-n-2(\theta+1)} \left( \int_{B(\xi, \tau/2)} |g(z) - g(y)| dy \right)^2 dz + A \\
& \lesssim r^n \int_{B(\xi, \tau/2)} \int_{B(\xi, \tau/2)} \frac{|g(z) - g(y)|^2}{|z-y|^{n+2(\theta+1)}} dz dy + A \\
& \lesssim \iint_{B(\xi, 4\tau)^2} \frac{|g(z) - g(y)|^2}{|z-y|^{n+2(\theta+1)}} dz dy,
\end{aligned}$$

where

$$A \lesssim \iint_{B(\xi, 4\tau)^2} \frac{|g(z) - g(y)|^2}{|z-y|^{n+2(\theta+1)}} dz dy,$$

and where we used  $|z-x| \gtrsim |z-y|$  on the region of integration.

Finally, we estimate

$$\begin{aligned}
& \int_{\mathbb{R}^n} \int_{B(\xi, \tau/2)^c} \frac{|(g-c)\chi_1(z) - (g-c)\chi_1(x)|^2}{|z-x|^{n+2(\theta+1)}} dz dx \\
& \lesssim \int_{B(\xi, 4\tau)} \int_{A(\xi, \tau/2, 4\tau)} \frac{|(g-c)(x)(\chi_1-1)(z) - (g-c)(x)(\chi_1-1)(x)|^2}{|z-x|^{n+2(\theta+1)}} dz dx + A \\
& \lesssim_{\nabla \xi} \int_{B(\xi, 4\tau)} \int_{A(\xi, \tau/2, 4\tau)} |z-x|^{-n-2\theta} \left( \int_{B(\xi, \tau/2)} |g(x) - g(y)| dy \right)^2 dz dx + A \\
& \lesssim \int_{B(\xi, 4\tau)} r^{-2\theta} \left( \int_{B(\xi, \tau/2)} |g(x) - g(y)| dy \right)^2 dx + A \\
& \lesssim \iint_{B(\xi, 4\tau)^2} \frac{|g(x) - g(y)|^2}{|x-y|^{n+2(\theta+1)}} dx dy
\end{aligned}$$

with  $A$  as before, using  $r \gtrsim |x-y|$  and  $|x-y|^{n+2\theta} \gtrsim |x-y|^{n+2(\theta+1)}$  on the region of integration.

Now we will handle the remaining  $\chi_j$  terms. For  $j \geq 2$ , by local coercivity (Lemma 6.2.5), the equality

$$BD(I + itBD)^{-1} = (I - (I + itBD)^{-1})/it,$$

and off-diagonal estimates of the families  $(I + itBD)^{-1}$  and  $(I + itBD)^{-2}$  of order  $M$ , we can estimate

$$\begin{aligned}
& \iint_{T(B(\xi, \tau))} |t^{-\theta}(I + itDB)^{-2}D(g - c)\chi_j(x)|^2 \frac{dx dt}{t} \\
& \lesssim \int_0^\tau t^{-2\theta} \int_{B(\xi, \tau)} |D(I + itBD)^{-2}(g - c)\chi_j(x)|^2 dx \frac{dt}{t} \\
& \lesssim \int_0^\tau t^{-2\theta} \left[ \int_{B(\xi, 2\tau)} |(BD(I + itBD)^{-2}(g - c)\chi_j(x)|^2 dx \right. \\
& \quad \left. + \tau^{-2} \int_{B(\xi, 2\tau)} |(I + itBD)^{-2}(g - c)\chi_j(x)|^2 dx \right] \frac{dt}{t} \\
& \lesssim \int_0^\tau t^{-2\theta} \left( \frac{2^j \tau}{t} \right)^{-2M} (t^{-2} + \tau^{-2}) \|(g - c)\chi_j\|_2^2 \frac{dt}{t} \\
& \lesssim \frac{2^{-2jM}}{\tau^{2(1+\theta)}} \|(g - c)\chi_j\|_2^2.
\end{aligned}$$

Furthermore, for each  $j \geq 2$  we have

$$\begin{aligned}
\|(g - c)\chi_j\|_2^2 & \leq \int_{A(\xi, 2^{j-1}\tau, 2^{j+2}\tau)} \left( \int_{B(\xi, \tau/2)} |g(x) - g(y)| dy \right)^2 dx \\
& \leq \tau^{-n} \int_{B(\xi, \tau/2)} \int_{A(\xi, 2^{j-1}\tau, 2^{j+2}\tau)} |g(x) - g(y)|^2 dx dy \\
& \lesssim \tau^{2(1-\theta)} 2^{j(n+2(1+\theta))} \int_{B(\xi, 4\tau)} \int_{A(\xi, 2^{j-1}\tau, 2^{j+2}\tau)} \frac{|g(x) - g(y)|^2}{|x - y|^{n+2(1+\theta)}} dx dy.
\end{aligned}$$

Putting these estimates together completes the proof of the lemma.  $\square$

*Proof of Theorem 6.2.12. Step 1: Reduction to a resolvent estimate.*

As stated in [15, Proof of Lemma 15.1], there exists  $\rho \in H^\infty$  of the form

$$\rho(z) = \sum_{m=1}^N c_m (1 + imz)^{-2}$$

for some scalars  $c_1, \dots, c_N \in \mathbb{C}$ , and  $\psi \in \Psi_N^2$  nondegenerate, such that

$$e^{-z} = \rho(z) + \psi(z) \quad \text{for all } z \in S_\mu^+.$$

We thus have

$$\|C_{DB}^+ f\|_{X^{\mathbf{p}}} \lesssim_N \|t \mapsto (I + itDB)^{-2} \chi^+(DB)f\|_{X^{\mathbf{p}}} + \|\mathbb{Q}_{\psi, DB} \chi^+(DB)f\|_{X^{\mathbf{p}}}. \quad (6.19)$$

For  $N$  sufficiently large we have

$$\psi \in \Psi_N^2 \subset \Psi_{(\theta(\mathbf{p})+n|\frac{1}{2}-j(\mathbf{p})|)+}^{-\theta(\mathbf{p})+} \cap \Psi_+^+ \subset \Psi(\mathbb{X}_{DB}^{\mathbf{p}}),$$

and so

$$\left\| \mathbb{Q}_{\psi, DB} \chi^+(DB) f \right\|_{X^{\mathbf{p}}} \lesssim \|f\|_{\mathbb{X}_{DB}^{\mathbf{p}}} \simeq \|f\|_{\mathbb{X}_D^{\mathbf{p}}}$$

by Proposition 6.2.7 and  $\mathbf{p}^\heartsuit \in I(\mathbf{X}, DB^*)$ . Therefore it suffices to prove the estimate

$$\left\| t \mapsto (I + itDB)^{-2} f \right\|_{X^{\mathbf{p}}} \lesssim \|f\|_{\mathbb{X}_D^{\mathbf{p}}} \quad (6.20)$$

for all  $f \in \overline{\mathcal{R}(DB)}$ . Applying this inequality to  $\chi^+(DB)f$  and invoking the boundedness of  $\chi^+(DB)$  on  $\mathbb{X}_{DB}^{\mathbf{p}}$  will yield

$$\left\| C_{DB}^+ f \right\|_{X^{\mathbf{p}}} \lesssim \|f\|_{\mathbb{X}_{DB}^{\mathbf{p}}} \simeq \|f\|_{\mathbb{X}_D^{\mathbf{p}}}.$$

To prove (6.20), by density (Corollary 6.1.7 and density of  $\mathcal{R}(D)$  in  $\mathbb{X}_D^2$ ),<sup>8</sup> it suffices to consider  $f = Dg$  for  $g \in \mathcal{D}(D) \cap \overline{\mathcal{R}(D)}$  such that

$$\|f\|_{\mathbb{X}_D^{\mathbf{p}}} \simeq \|f\|_{X^{\mathbf{p}}} \simeq \|g\|_{X^{\mathbf{p}+1}}$$

as in Lemma 6.2.2.

**Step 2a: Completing the proof for Hardy–Sobolev spaces.**

Suppose  $i(\mathbf{p}) < \infty$  and  $(X, \mathbb{X}) = (T, \mathbb{H})$ . Lemma 6.2.15 and a crude estimate give

$$\begin{aligned} & \iint_{T(B(\xi, \tau))} |t^{-\theta(\mathbf{p})} (I + itDB)^{-2} f(x)|^2 \frac{dx dt}{t} \\ & \lesssim_M \left( 1 + \sum_{j=2}^{\infty} 2^{-2j(M - \frac{n}{2} - (1 + \theta(\mathbf{p})))} \right) \int_{B(\xi, 4\tau)} |\mathcal{D}_{1+\theta(\mathbf{p})}^2 g(x)|^2 dx, \end{aligned}$$

and so by taking  $M > \frac{n}{2} - (1 + \theta(\mathbf{p}))$  we get

$$\iint_{T(B(\xi, \tau))} |t^{-\theta(\mathbf{p})} (I + itDB)^{-2} f(x)|^2 \frac{dx dt}{t} \lesssim \int_{B(\xi, 4\tau)} |\mathcal{D}_{1+\theta(\mathbf{p})}^2 g(x)|^2 dx.$$

Hence for all  $\xi \in \mathbb{R}^n$  we have

$$\mathcal{C}(t \mapsto t^{-\theta(\mathbf{p})} (I + itDB)^{-2} f)(\xi)^2 \lesssim \sup_{\tau > 0} \int_{B(\xi, 4\tau)} |\mathcal{D}_{1+\theta(\mathbf{p})}^2 g|^2 = \mathcal{M}_2(\mathcal{D}_{1+\theta(\mathbf{p})}^2 g)(\xi)^2,$$

and so by Theorem 5.1.9, boundedness of  $\mathcal{M}_2$  on  $L^{i(\mathbf{p})}$  (since  $i(\mathbf{p}) > 2$ ), Lemma 5.1.49 (using  $1 + \theta(\mathbf{p}) \in (0, 1)$ ),  $Dg = f$ , and  $\mathbf{p} \in I(\mathbf{H}, DB)$ , we get

$$\begin{aligned} \left\| t \mapsto (I + itDB)^{-2} f \right\|_{T^{\mathbf{p}}} & \lesssim \left\| \mathcal{M}_2(\mathcal{D}_{1+\theta(\mathbf{p})}^2 g) \right\|_{L^{i(\mathbf{p})}} \\ & \lesssim \left\| \mathcal{D}_{1+\theta(\mathbf{p})}^2 g \right\|_{L^{i(\mathbf{p})}} \\ & \simeq \|g\|_{\dot{H}^{\mathbf{p}+1}} \\ & \simeq \|f\|_{\mathbb{H}_D^{\mathbf{p}}}, \end{aligned}$$

---

<sup>8</sup>When  $\mathbf{p}$  is infinite, we use weak-star density.

which completes the proof in the Hardy–Sobolev case.

**Step 2b: Completing the proof for BMO-Sobolev spaces.** Suppose  $\mathbf{p} = (\infty, \theta; 0)$  and  $(X, \mathbb{X}) = (T, \mathbb{H})$ . For all  $(t, x) \in \mathbb{R}_+^{1+n}$ , Lemma 6.2.15 and the Strichartz characterisation of  $BMO_{1+\theta}$  (Theorem 5.1.51) yield

$$\begin{aligned} & \left( t^{-n} \iint_{T(B(x,t))} |\tau^{-\theta} (I + i\tau DB)^{-2} f(\xi)|^2 d\xi \frac{d\tau}{\tau} \right)^{1/2} \\ & \lesssim_M t^{-n/2} \left( t^n \|g\|_{BMO_{1+\theta}}^2 + \sum_{j=2}^{\infty} 2^{-2j(M-\frac{n}{2}-(1+\theta))} (2^{j+2}t)^n \|g\|_{BMO_{1+\theta}}^2 \right)^{1/2} \\ & \simeq \|g\|_{BMO_{1+\theta}} \left( 1 + \sum_{j=2}^{\infty} 2^{-2j(M-n-(1+\theta))} \right)^{1/2} \\ & \simeq \|g\|_{BMO_{1+\theta}} \end{aligned}$$

provided that  $M$  is sufficiently large. Therefore we have as in the previous step

$$\left\| \tau \mapsto (I + i\tau DB)^{-2} f \right\|_{T\mathbf{p}} \lesssim \|g\|_{\mathbf{H}^{\mathbf{p}+1}} \simeq \|f\|_{\mathbb{H}_D^{\mathbf{p}}},$$

which completes the proof in the BMO-Sobolev case.

**Step 2c: Completing the proof for Hölder spaces.** Let  $\mathbf{p} = (\infty, \theta; \alpha)$ . First we prove the result for  $X = T$ . By the definition of the Hölder norm we have

$$G(x, y) \leq \|g\|_{\dot{\Lambda}_{1+\theta+\alpha}}^2 |x - y|^{2\alpha-n},$$

and so by Lemma 6.2.15,

$$\begin{aligned} & t^{-\alpha} \left( t^{-n} \iint_{T(B(x,t))} |\tau^{-\theta} (I + i\tau DB)^{-2} f(\xi)|^2 d\xi \frac{d\tau}{\tau} \right)^{1/2} \\ & \lesssim_M t^{-\alpha-\frac{n}{2}} \|g\|_{\dot{\Lambda}_{1+\theta+\alpha}} \left( \iint_{B(x,4t)^2} \frac{d\xi d\eta}{|\xi - \eta|^{n-2\alpha}} \right. \\ & \quad \left. + \sum_{j=2}^{\infty} 2^{-2j(M-\frac{n}{2}-(1+\theta))} \int_{B(x,4t)} \int_{A(x,2^{j-1}t,2^{j+2}t)} \frac{d\xi d\eta}{|\xi - \eta|^{n-2\alpha}} \right)^{1/2} \\ & \lesssim t^{-\alpha-\frac{n}{2}} \|g\|_{\dot{\Lambda}_{1+\theta+\alpha}} \left( t^{n+2\alpha} + \sum_{j=2}^{\infty} 2^{-2j(M-\frac{n}{2}-(1+\theta))} 2^{-j(n-2\alpha)} t^{n+2\alpha} \right)^{1/2} \\ & = \|g\|_{\dot{\Lambda}_{1+\theta+\alpha}} \end{aligned}$$

for  $M$  sufficiently large. Therefore, by the same concluding argument as in the previous steps,

$$\left\| \tau \mapsto (I + i\tau DB)^{-2} f \right\|_{T\mathbf{p}} \lesssim \|f\|_{\dot{\Lambda}^{\theta+\alpha}}.$$

In the case that  $X = Z$ , since  $\mathbf{p}$  is infinite, Lemma 5.1.34 yields  $T^{\mathbf{p}} \hookrightarrow Z^{\mathbf{p}}$ , and so by previous estimate we have

$$\left\| \tau \mapsto (I + itDB)^{-2} f \right\|_{Z^{\mathbf{p}}} \lesssim \left\| \tau \mapsto (I + itDB)^{-2} f \right\|_{T^{\mathbf{p}}} \lesssim \|f\|_{\dot{A}^{\theta+\alpha}}.$$

This completes the proof in the Hölder space case.

**Step 2d: Completing the proof for Besov spaces.**

Let  $\mathbf{p} = (p, \theta)$ . We use a slightly different argument here. Fix cutoff functions  $\chi_1, \chi \in C_c^\infty(\mathbb{R}^n)$  with

$$\begin{aligned} \text{supp } \chi_1 &\subset B(0, 4), & \chi_1|_{B(0,2)} &\equiv \text{const}, \\ \text{supp } \chi &\subset A(0, 1/2, 4) & \chi|_{A(0,1,2)} &\equiv \text{const}, \end{aligned}$$

for all integers  $j \geq 2$  define  $\chi_j(x) := \chi(2^{-j}x)$ , and for all  $j \geq 1$  define

$$\eta_j(t, x, \xi) := \chi_j\left(\frac{x - \xi}{t}\right) \quad ((t, x) \in \mathbb{R}_+^{1+n}, \xi \in \mathbb{R}^n);$$

as before, these functions can be chosen such that  $\sum_{j=1}^\infty \eta_j = 1$ . Also define

$$\tilde{g}(t, x, \xi) := g(\xi) - \int_{B(x,t)} g(\zeta) d\zeta \quad ((t, x) \in \mathbb{R}_+^{1+n}, \xi \in \mathbb{R}^n).$$

By the triangle inequality we have

$$\begin{aligned} &\left\| t \mapsto (I + itDB)^{-2} f \right\|_{Z_\theta^p}^p \\ &\lesssim \sum_{j=1}^\infty \iint_{\mathbb{R}_+^{1+n}} \left( \iint_{\Omega(t,x)} |\tau^{-\theta} (I + i\tau DB)^{-2} D(\tilde{g}\eta_j(t, x, \xi))|^2 d\xi d\tau \right)^{p/2} dx \frac{dt}{t}, \end{aligned} \quad (6.21)$$

where the operators involving  $D$  and  $B$  act in the  $\xi$  variable. By using local coercivity (Lemma 6.2.5) as in the proof of Lemma 6.2.15, the  $j$ -th term in (6.21) can be estimated by

$$\begin{aligned} &\iint_{\mathbb{R}_+^{1+n}} \left( \iint_{\Omega(t,x)} |\tau^{-\theta} (I + i\tau DB)^{-2} D(\tilde{g}\eta_j(t, x, \xi))|^2 d\xi d\tau \right)^{p/2} dx \frac{dt}{t} \\ &\lesssim \iint_{\mathbb{R}_+^{1+n}} \left( \iint_{\Omega_c(t,x)} |\tau^{-\theta-1} (I + i\tau BD)^{-1} (\tilde{g}\eta_j(t, x, \xi))|^2 d\xi d\tau \right)^{p/2} dx \frac{dt}{t} \\ &+ \iint_{\mathbb{R}_+^{1+n}} \left( \iint_{\Omega_c(t,x)} |\tau^{-\theta-1} (I + i\tau BD)^{-2} (\tilde{g}\eta_j(t, x, \xi))|^2 d\xi d\tau \right)^{p/2} dx \frac{dt}{t} \end{aligned}$$

with Whitney parameter  $c = (2, 2)$ . The two terms in this sum differ only in the power of the resolvent. The resolvent families  $(I + i\tau DB)^{-1}$  and  $(I + i\tau DB)^{-2}$

both satisfy off-diagonal estimates of arbitrarily large order  $M$  (as off-diagonal estimates may be composed); we will use this to estimate the terms above, making reference only to  $(I + i\tau DB)^{-1}$ .

For  $j = 1$  we estimate

$$\begin{aligned}
& \iint_{\mathbb{R}_+^{1+n}} \left( \iint_{\Omega_c(t,x)} |\tau^{-\theta-1}(I + i\tau BD)^{-1}(\tilde{g}\eta_1(t, x, \xi))|^2 d\xi d\tau \right)^{p/2} dx \frac{dt}{t} \\
& \lesssim \iint_{\mathbb{R}_+^{1+n}} \left( t^{-2\theta-2-n} \int_{B(x,4t)} |\tilde{g}(t, x, \xi)|^2 d\xi \right)^{p/2} dx \frac{dt}{t} \\
& \lesssim \iint_{\mathbb{R}_+^{1+n}} \left( \int_{B(x,4t)} \int_{B(x,t)} \left| \frac{g(\xi) - g(\zeta)}{t^{\theta+1}} \right|^2 d\zeta d\xi \right)^{p/2} dx \frac{dt}{t} \\
& \leq \iint_{\mathbb{R}_+^{1+n}} \int_{B(x,4t)} \int_{B(x,t)} \left| \frac{g(\xi) - g(\zeta)}{t^{\theta+1}} \right|^p d\zeta d\xi dx \frac{dt}{t} \\
& = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_0^\infty \int_{B(\xi,4t) \cap B(\zeta,t)} dx \frac{1}{t^{2n+p(\theta+1)}} \frac{dt}{t} |g(\xi) - g(\zeta)|^p d\zeta d\xi \\
& \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{|\zeta-\xi|/5}^\infty \frac{1}{t^{n+p(\theta+1)}} \frac{dt}{t} |g(\xi) - g(\zeta)|^p d\zeta d\xi \\
& \simeq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|g(\xi) - g(\zeta)|^p}{|\zeta - \xi|^{n+p(\theta+1)}} d\zeta d\xi \\
& \simeq \|g\|_{\dot{B}_{\theta+1}^{p,p}} \\
& \simeq \|f\|_{\dot{B}_\theta^{p,p}},
\end{aligned}$$

using that  $\eta_1(t, x, \cdot)$  is supported in  $B(x, 4t)$ , that  $p/2 > 1$ , that  $B(\xi, 4t) \cap B(\zeta, t)$  is nonempty only if  $t > |\zeta - \xi|/5$ , and the Besov norm characterisation from Theorem 5.1.50.

For  $j \geq 2$  we have, using off-diagonal estimates,

$$\begin{aligned}
& \iint_{\mathbb{R}_+^{1+n}} \left( \iint_{\Omega_c(t,x)} |\tau^{-\theta-1} (I + i\tau BD)^{-1} (\tilde{g}\eta_j(t, x, \xi))|^2 d\xi d\tau \right)^{p/2} dx \frac{dt}{t} \\
& \lesssim 2^{-j(Mp-(np)/2)} \iint_{\mathbb{R}_+^{1+n}} \left( t^{-2\theta-2} \int_{A(x, 2^{j-1}t, 2^{j+2}t)} \int_{B(x,t)} |g(\xi) - g(\zeta)|^2 d\zeta d\xi \right)^{p/2} dx \frac{dt}{t} \\
& \leq 2^{-j(Mp-(np)/2)} \iint_{\mathbb{R}_+^{1+n}} \int_{A(x, 2^{j-1}t, 2^{j+2}t)} \int_{B(x,t)} \left| \frac{g(\xi) - g(\zeta)}{t^{\theta+1}} \right|^p d\zeta d\xi dx \frac{dt}{t} \\
& = 2^{-j(Mp-(np)/2+n)} \cdot \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_0^\infty \frac{1}{t^{2n+p(\theta+1)}} \int_{B(\zeta,t) \cap A(\xi, 2^{j-1}t, 2^{j+1}t)} dx \frac{dt}{t} |g(\xi) - g(\zeta)|^p d\xi d\zeta \\
& \lesssim 2^{-j(Mp-(np)/2+2n)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{2^{-j}|\zeta-\xi|}^\infty \frac{1}{t^{n+p(\theta+1)}} \frac{dt}{t} |g(\xi) - g(\zeta)|^p d\xi d\zeta \\
& \simeq 2^{-j(pM-(np)/2+n-p(\theta+1))} \|f\|_{\dot{B}_\theta^{p,p}}
\end{aligned}$$

arguing similarly to before. For  $M$  sufficiently large, we can thus estimate (6.21) by summing a geometric series, yielding

$$\left\| t \mapsto (I + itDB)^{-2} f \right\|_{Z_\theta^p}^p \lesssim \|f\|_{\dot{B}_\theta^{p,p}}^p$$

as required. This completes the proof.  $\square$



# Chapter 7

## Elliptic equations, Cauchy–Riemann systems, and boundary value problems

In this section we implicitly work with a fixed  $m \in \mathbb{N}$ , meaning that we consider  $L_A u = 0$  with  $u$  a  $\mathbb{C}^m$ -valued function. All of our arguments are independent of  $m$ . As in the previous section, we fix the Dirac operator  $D$  and multipliers  $B$  from Subsection 4.1.2.

### 7.1 Basic properties of solutions

We will use the following properties of conormal gradients of solutions to  $L_A u = 0$  (or equivalently, of solutions to  $(\text{CR})_{D\hat{A}}$ ; see Theorem 4.1.3 in the introduction).

**Proposition 7.1.1.** *Suppose that  $u$  solves  $L_A u = 0$ . Then the following are true.*

- (1) *The transversal derivative  $\partial_t u$  solves  $L_A(\partial_t u) = 0$ .*
- (2) *The function  $t \mapsto \nabla_A u(t, \cdot)$  is in  $C^\infty(\mathbb{R}_+ : L^2_{loc}(\mathbb{R}^n))$ , and for all Whitney parameters  $c = (c_0, c_1)$  and  $t \in \mathbb{R}_+$  we have*

$$\int_{B(x, c_0 t)} |\nabla_A u(t, x)|^2 dx \lesssim \iint_{\Omega_c(t, x)} |\nabla_A u(s, y)|^2 ds dy.$$

(3) For all exponents  $\mathbf{p}$ , all  $k \in \mathbb{N}$ , and all  $C \geq 1$  we have

$$\begin{aligned} \sup_{\substack{t, t' \in \mathbb{R}_+ \\ C^{-1} \leq t/t' \leq C}} \left\| \partial_t^k \nabla_A u(t, \cdot) \right\|_{E^{\mathbf{p}-k}(t')} &\lesssim_C \left\| \partial_t^k \nabla_A u \right\|_{X^{\mathbf{p}-k}} \\ &= \left\| \nabla_A \partial_t^k u \right\|_{X^{\mathbf{p}-k}} \\ &\lesssim \left\| \nabla_A u \right\|_{X^{\mathbf{p}}} . \end{aligned}$$

In particular, if  $\nabla_A u$  is in  $X^{\mathbf{p}}$ , then the function  $t \mapsto \nabla_A u(t, \cdot)$  is in  $C^\infty(\mathbb{R}_+ : E^{\mathbf{p}})$

*Proof.* (1) follows from  $t$ -independence of the coefficients. The remaining statements are consequences of the classical Caccioppoli inequality, and are proven in [15, §5] for tent spaces. The corresponding  $Z$ -space statements are proven in the same way.  $\square$

*Remark 7.1.2.* By Theorem 4.1.3, if  $F$  is a solution to the Cauchy–Riemann system (4.6), then parts (2) and (3) of Proposition 7.1.1 hold with  $\nabla_A u$  replaced by  $F$ .

Furthermore, suppose that  $G$  solves the anti-Cauchy–Riemann system

$$(a \text{ CR})_{DB} : \begin{cases} \partial_t G - DBG = 0 & \text{in } \mathbb{R}_+^{1+n}, \\ \text{curl}_{\parallel} G_{\parallel} = 0 & \text{in } \mathbb{R}_+^{1+n} \end{cases} \quad (7.1)$$

defined analogously to  $(\text{CR})_{DB}$  but with a sign change. Then the reflection  $F(t) := G(-t)$  solves  $(\text{CR})_{DB}$  on the lower half-space  $\mathbb{R}_-^{1+n}$ . By using  $X$ -spaces associated with the lower half-space rather than the upper half-space, parts (2) and (3) of Proposition 7.1.1 hold with  $\nabla_A u$  replaced by  $F$  and with  $\mathbb{R}_+$  replaced by  $\mathbb{R}_-$ . A simple reflection argument then shows parts (2) and (3) of Proposition 7.1.1 hold for  $G$ .

The following technical lemma is analogous to [15, Lemma 10.2].

**Lemma 7.1.3.** Fix  $\mathbf{p}$  with  $i(\mathbf{p}) < 2$  and  $\theta(\mathbf{p}) < 0$ , suppose  $M \in \mathbb{N}$ , and let  $f \in \mathbb{X}_{DB}^{\mathbf{p}}$ . Then for all  $t > 0$  we have that  $(tDB)^M e^{-t[DB]} \chi^\pm(DB) f \in E^{\mathbf{p}}$ , with

$$\sup_{t>0} \left\| (tDB)^M e^{-t[DB]} \chi^\pm(DB) f \right\|_{E^{\mathbf{p}}(t)} \lesssim \|f\|_{\mathbb{X}_{DB}^{\mathbf{p}}} .$$

*Proof.* We estimate

$$\begin{aligned} \sup_{t>0} \left\| (tDB)^M e^{-t[DB]} \chi^\pm(DB) f \right\|_{E^{\mathbf{p}}(t)} &\lesssim \left\| t \mapsto (tDB)^M e^{-t[DB]} \chi^\pm(DB) f \right\|_{X^{\mathbf{p}}} \\ &\simeq \|f\|_{\mathbb{X}_{DB}^{\mathbf{p}}} . \end{aligned}$$

The first line comes from Proposition 7.1.1, using that  $(DB)^M e^{-t[DB]} \chi^\pm (DB) f$  solves either  $(\text{CR})_{DB}$  or  $(a \text{ CR})_{DB}$ . The second line is due to the fact that  $[z \mapsto z^M e^{-[z]}] \in \Psi(\mathbb{X}_{DB}^{\mathbf{p}})$  when  $i(\mathbf{p}) < 2$  and  $\theta(\mathbf{p}) < 0$ .  $\square$

## 7.2 Decay of solutions at infinity

In the boundary value problems introduced in Subsection 4.1.1, we have imposed the decay condition

$$\lim_{t \rightarrow \infty} \nabla_{\parallel} u(t, \cdot) = 0 \quad \text{in } \mathcal{Z}'(\mathbb{R}^n : \mathbb{C}^{mn})$$

for a solution  $u$  to  $L_A u$  with  $\nabla u$  in  $X^{\mathbf{p}}$ . In this section we will show that this condition is redundant for certain  $\mathbf{p}$  (quantified in terms of  $A$ ). In fact, our results give not just decay in  $\mathcal{Z}'$ , but in the slice space  $E^\infty$  (in the setting of Lemma 7.2.1) or in  $L^2$  (in Lemma 7.2.4).<sup>1</sup>

Classical elliptic theory implies that there exists a number  $\lambda(A) \in (0, n+1)$  such that for all  $\lambda \in [0, \lambda(A))$ , for all  $(t_0, x_0) \in \mathbb{R}_+^{n+1}$  and  $0 < r < R < \infty$ , and for all weak solutions  $u$  to  $L_A u = 0$ , we have

$$\iint_{B((t_0, x_0), r)} |\nabla u(t, x)|^2 dx dt \lesssim_\lambda \left(\frac{r}{R}\right)^\lambda \iint_{B((t_0, x_0), R)} |\nabla u(t, x)|^2 dx dt, \quad (7.2)$$

where  $B((t_0, x_0), r)$  and  $B((t_0, x_0), R)$  denote open balls in  $\mathbb{R}^{1+n}$ , with  $B((t_0, x_0), R)$  contained in  $\mathbb{R}_+^{1+n}$ . These balls can be taken with respect to any norm on  $\mathbb{R}^{1+n}$ , keeping in mind that the implicit constant in (7.2) will depend on the chosen norm. By ellipticity we may replace the gradient  $\nabla$  with the conormal gradient  $\nabla_A$  in (7.2).

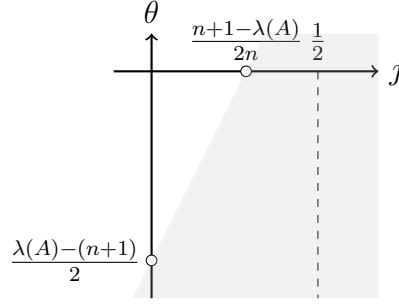
**Lemma 7.2.1.** *Suppose that the exponent  $\mathbf{p}$  lies in the shaded region pictured in Figure 7.1, which depends on  $\lambda(A)$ . Let  $u$  be a solution to  $L_A u = 0$  on  $\mathbb{R}_+^{1+n}$  such that  $\nabla_A u \in X^{\mathbf{p}}$ . Then  $\lim_{t \rightarrow \infty} \nabla_A u(t, \cdot) = 0$  in  $E^\infty$  (and therefore also in  $\mathcal{Z}'$ ).*

*Remark 7.2.2.* The shaded region in Figure 7.1 is the open half-plane determined by the equation  $j(\mathbf{p}) > \frac{\theta(\mathbf{p})}{n} - \frac{n+1-\lambda(A)}{2n}$ . Note that  $\frac{n+1-\lambda(A)}{2n} \geq \frac{1}{2}$  when  $\lambda(A) \leq 1$ . In Lemma 7.2.4 we will handle exponents  $\mathbf{p}$  with  $i(\mathbf{p}) \leq 2$  and  $\theta(\mathbf{p}) < 0$  independently of  $\lambda(A)$ .

---

<sup>1</sup>Demanding decay in  $\mathcal{Z}'$  is really just an artefact of having identified the classical smoothness spaces  $\mathbf{X}^{\mathbf{p}}$  as subspaces of  $\mathcal{Z}'$ .

Figure 7.1: The exponent region in Lemma 7.2.1.



*Proof.* The region pictured in Figure 7.1 is precisely the set of exponents  $\mathbf{p}$  such that there exists an infinite exponent  $\mathbf{q}$  with  $\mathbf{p} \hookrightarrow \mathbf{q}$  and  $r(\mathbf{q}) < \frac{\lambda(A) - (n+1)}{2}$ . Fix such a  $\mathbf{q}$ . For all  $\lambda < \lambda(A)$  we can estimate

$$\begin{aligned} \|\nabla_A u(t, \cdot)\|_{E_0^\infty(1)} &\simeq \sup_{x \in \mathbb{R}^n} \left( \int_{B(x,1)} |\nabla_A u(t, y)|^2 dy \right)^{1/2} \\ &\lesssim \sup_{x \in \mathbb{R}^n} \left( \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} \int_{B(x,1)} |\nabla_A u(s, y)|^2 dy ds \right)^{1/2} \end{aligned} \quad (7.3)$$

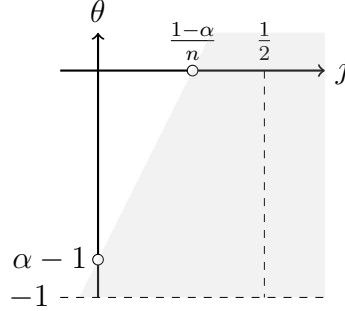
$$\begin{aligned} &\lesssim \sup_{x \in \mathbb{R}^n} \left( t^{-\lambda} \int_{t-\frac{t}{2}}^{t+\frac{t}{2}} \int_{B(x,t)} |\nabla_A u(s, y)|^2 dy ds \right)^{1/2} \quad (7.4) \\ &\lesssim t^{\frac{(n+1)-\lambda}{2} + r(\mathbf{q})} \left( \int_{t-\frac{t}{2}}^{t+\frac{t}{2}} \|\nabla_A u(s, \cdot)\|_{E^{\mathbf{q}}(s)}^2 ds \right)^{1/2} \\ &\lesssim t^{\frac{(n+1)-\lambda}{2} + r(\mathbf{q})} \|\nabla_A u\|_{X^{\mathbf{p}}} \end{aligned}$$

where (7.3) follows from Proposition 7.1.1, (7.4) follows from (7.2),<sup>2</sup> and the last line follows from the embeddings  $E^{\mathbf{p}}(s) \hookrightarrow E^{\mathbf{q}}(s)$  and another application of Proposition 7.1.1. For  $\lambda$  sufficiently close to  $\lambda(A)$  we have  $(n+1-\lambda)/2 + r(\mathbf{q}) < 0$ , and so we find that  $\lim_{t \rightarrow \infty} \nabla_A u(t, \cdot) = 0$  in  $E^\infty$ .  $\square$

*Remark 7.2.3.* It is known that  $\lambda(A) > n - 1$  if and only if  $A$  satisfies the De Giorgi–Nash–Moser condition (7.51) of all exponents less than  $\alpha = (\lambda(A) - (n - 1))/2$ . In this case we have  $\frac{\lambda(A) - (n+1)}{2} = \alpha - 1$  and  $\frac{n+1-\lambda(A)}{2n} = \frac{1-\alpha}{n}$ , and Lemma 7.2.1 then holds for the shaded region pictured in Figure 7.2. Evidently this region increases as the De Giorgi–Nash–Moser exponent  $\alpha$  increases.

<sup>2</sup>Here we use the balls  $B((t, x), r) := (t - r/2, t + r/2) \times B(x, r)$ .

Figure 7.2: The region in Lemma 7.2.1 in the case that  $A$  satisfies the De Giorgi–Nash–Moser condition with exponent  $\alpha$ .



A different argument can be used to deduce decay in  $L^2$  for exponents  $\mathbf{p}$  with  $i(\mathbf{p}) \leq 2$  and  $\theta(\mathbf{p}) < 0$ .

**Lemma 7.2.4.** *Let  $\mathbf{p} = (p, s)$  with  $i(\mathbf{p}) \leq (0, 2]$  and  $\theta(\mathbf{p}) < 0$ , and suppose  $F \in X^{\mathbf{p}}$  solves  $(\text{CR})_{DB}$  or  $(a\text{CR})_{DB}$ . Then  $\lim_{t \rightarrow 0} F(t) = 0$  in  $L^2$ .*

*Proof.* By Proposition 7.1.1, for all  $t \in \mathbb{R}_+$  we have

$$\|F(t)\|_{E^{\mathbf{p}}(t)} \lesssim \|F\|_{X^{\mathbf{p}}},$$

and so

$$\|F(t)\|_{L^2} \lesssim \|F(t)\|_{E_0^{i(\mathbf{p})}(t)} = t^{\theta(\mathbf{p})} \|F(t)\|_{E^{\mathbf{p}}(t)} \lesssim t^{\theta(\mathbf{p})} \|F\|_{X^{\mathbf{p}}}$$

using the embedding  $E_0^{i(\mathbf{p})}(t) \hookrightarrow E_0^2(t) = L^2$ .<sup>3</sup> Since  $\theta(\mathbf{p}) < 0$ , we have

$$\lim_{t \rightarrow \infty} F(t) = 0$$

in  $L^2$ . □

## 7.3 Classification of solutions to Cauchy–Riemann systems

In this section we will prove the following classification theorems for solutions to  $(\text{CR})_{DB}$  (as formulated in Subsection 4.1.2 of the introduction).

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<sup>3</sup>The equality  $E_0^2(t) = L^2$  is a consequence of Fubini’s theorem.

**Theorem 7.3.1** (Classification of solutions to  $(\text{CR})_{DB}$ ,  $i(\mathbf{p}) \leq 2$ ). *Let  $\mathbf{p} = (p, s)$  with  $p \leq 2$  and  $s < 0$ , and fix a completion  $\mathbf{X}_{DB}^{\mathbf{p}}$  of  $\mathbb{X}_{DB}^{\mathbf{p}}$ .*

- (i) *For all  $F_0 \in \mathbf{X}_{DB}^{\mathbf{p},+}$ ,  $\mathbf{C}_{DB}^+ F_0$  solves  $(\text{CR})_{DB}$ , and  $\|\mathbf{C}_{DB}^+ F_0\|_{X^{\mathbf{p}}} \lesssim \|F_0\|_{\mathbf{X}_{DB}^{\mathbf{p}}}$ .*
- (ii) *Conversely, if  $F \in X^{\mathbf{p}}$  solves  $(\text{CR})_{DB}$ , then there exists a unique  $F_0 \in \mathbf{X}_{DB}^{\mathbf{p},+}$  such that  $F = \mathbf{C}_{DB}^+ F_0$ . Furthermore,  $\|F_0\|_{\mathbf{X}_{DB}^{\mathbf{p}}} \lesssim \|F\|_{X^{\mathbf{p}}}$ .*

When  $p > 2$  the argument is much more complicated, we must restrict attention to exponents  $\mathbf{p}$  such that the adapted space  $\mathbb{X}_{DB}^{\mathbf{p},\heartsuit}$  may be identified with the classical space  $\mathbb{X}_D^{\mathbf{p},\heartsuit}$ , and we need an additional decay condition on  $F$ .

**Theorem 7.3.2** (Classification of solutions to  $(\text{CR})_{DB}$ ,  $i(\mathbf{p}) > 2$ ). *Let  $\mathbf{p}$  be an exponent with  $i(\mathbf{p}) > 2$  and  $\theta(\mathbf{p}) \in (-1, 0)$ , and such that  $\mathbf{p}^{\heartsuit} \in I(\mathbf{X}, DB^*)$ . In particular, for such  $\mathbf{p}$  we have identified  $\mathbf{X}_{DB}^{\mathbf{p},+}$  as a subspace of  $\mathbf{X}_D^{\mathbf{p}}$ .*

- (i) *If  $F_0 \in \mathbf{X}_{DB}^{\mathbf{p},+}$ , then  $\mathbf{C}_{DB}^+ F_0$  solves  $(\text{CR})_{DB}$ ,  $\lim_{t \rightarrow \infty} \mathbf{C}_{DB}^+ F_0(t)_{\parallel} = 0$  in  $\mathcal{Z}'(\mathbb{R}^n : \mathbb{C}^{nm})$ , and  $\|\mathbf{C}_{DB}^+ F_0\|_{X^{\mathbf{p}}} \lesssim \|F_0\|_{\mathbf{X}_{DB}^{\mathbf{p}}}$ .*
- (ii) *Conversely, if  $F \in X^{\mathbf{p}}$  solves  $(\text{CR})_{DB}$  and  $\lim_{t \rightarrow \infty} F(t)_{\parallel} = 0$  in  $\mathcal{Z}'(\mathbb{R}^n : \mathbb{C}^{nm})$ , then there exists a unique  $F_0 \in \mathbf{X}_{DB}^{\mathbf{p},+} = \mathbf{X}_D^{\mathbf{p}}$  such that  $F = \mathbf{C}_{DB}^+ F_0$ . Furthermore,  $\|F_0\|_{\mathbf{X}_{DB}^{\mathbf{p}}} \lesssim \|F\|_{X^{\mathbf{p}}}$ .*

Note that if  $\mathbf{p}$  is finite, then  $\mathbf{p}^{\heartsuit} \in I(\mathbf{X}, DB^*)$  if and only if  $\mathbf{p} \in I(\mathbf{X}, DB)$  (Proposition 6.2.7). Note also that if  $\mathbf{p}$  is in the region given by Lemma 7.2.1, then the decay condition on  $F$  is redundant. In particular, this holds for all  $\mathbf{p}$  as in Theorem 7.3.1, so the decay condition need not be included there.

### 7.3.1 Construction of solutions via Cauchy extension

Here we will prove part (i) of Theorems 7.3.1 and 7.3.2. We will deal with both theorems simultaneously

Let  $F_0 \in \mathbf{X}_{DB}^{\mathbf{p},+}$ . Then the estimate  $\|\mathbf{C}_{DB}^+ F_0\|_{X^{\mathbf{p}}} \lesssim \|F_0\|_{\mathbf{X}_{DB}^{\mathbf{p}}}$  follows from either Theorem 6.1.25 or Theorem 6.2.12.

In Proposition 6.1.24 we showed that  $\mathbf{C}_{DB}^+ F_0$  solves  $(\text{CR})_{DB}$  strongly in  $\mathbf{X}_{DB}^{\mathbf{p},+}$ . Generally  $\mathbf{X}_{DB}^{\mathbf{p},+}$  need not be contained in  $L_{\text{loc}}^2(\mathbb{R}^n)$ , and so these two solution concepts need not coincide. We must argue differently here. If  $F_0 \in \overline{\mathcal{R}(DB)}$ , then Proposition 5.2.6 implies that  $\mathbf{C}_{DB}^+ F_0$  solves  $(\text{CR})_{DB}$  strongly in  $C^\infty(\mathbb{R}_+ : L^2)$ , and this implies that  $\mathbf{C}_{DB}^+ F_0$  solves  $(\text{CR})_{DB}$ . It remains to deal with  $F_0 \in \mathbf{X}_{DB}^{\mathbf{p}} \setminus \overline{\mathcal{R}(DB)}$ . For such an  $F_0$ , let  $(F_0^k)_{k \in \mathbb{N}}$  be a sequence in  $\mathbb{X}_{DB}^{\mathbf{p}}$  which converges to  $F_0$  as  $k \rightarrow \infty$

(in the weak-star topology when  $\mathbf{p}$  is infinite). Then, again using either Theorem 6.1.25 or Theorem 6.2.12, we have

$$\lim_{k \rightarrow \infty} C_{DB}^+ F_0^k = C_{DB}^+ F_0 \quad \text{in } X^{\mathbf{p}},$$

and hence also in  $L_{\text{loc}}^2(\mathbb{R}_+^{1+n})$ . It follows that  $C_{DB}^+ F_0$  solves  $(\text{CR})_{DB}$ .

It remains to show that  $\lim_{t \rightarrow \infty} C_{DB}^+ F_0(t)_{\parallel} = 0$  in  $\mathcal{Z}'(\mathbb{R}^n : \mathbb{C}^{nm})$  when  $i(\mathbf{p}) > 2$ . This follows from Proposition 6.1.24, since we have  $\lim_{t \rightarrow \infty} C_{DB}^+ F_0(t) = 0$  in  $X_{DB}^{\mathbf{p}} \hookrightarrow \mathcal{Z}'$ .

### 7.3.2 Initial limiting arguments

We now begin preparation for the proof of part (ii) of Theorems 7.3.1 and 7.3.2. This section is a rephrasing of the start of [15, §8]. There are no fundamentally new ideas, but the notation and the flow of ideas are simplified.

For  $t_0 \in \mathbb{R}_+$  we write  $\mathbb{R}_{t_0} := \mathbb{R} \setminus \{t_0\}$  and  $\mathbb{R}_{+,t_0} := \mathbb{R}_+ \setminus \{t_0\}$ .

**Definition 7.3.3.** For  $t_0 \in \mathbb{R}_+$  and  $\varphi \in L^2(\mathbb{R}^n)$ , we define the test function  $\mathbf{G}_{t_0,\varphi} \in C^\infty(\mathbb{R}_{+,t_0} : \mathcal{D}(B^*D))$  by

$$\mathbf{G}_{t_0,\varphi}(t) := \text{sgn}(t_0 - t) e^{-[(t_0-t)B^*D]} \chi^{\text{sgn}(t_0-t)}(B^*D) \mathbb{P}_{\overline{\mathcal{R}(B^*D)}} \varphi$$

for all  $t \in \mathbb{R}_{+,t_0}$ .

Note that  $\partial_t \mathbf{G}_{t_0,\varphi} = B^*D \mathbf{G}_{t_0,\varphi}$ . Also observe that since  $D$  annihilates the nullspace  $\mathcal{N}_2(B^*D)$  and since  $L^2(\mathbb{R}^n) = \mathcal{N}_2(B^*D) \oplus \overline{\mathcal{R}(B^*D)}$ , whenever  $\varphi \in \mathcal{D}(D)$ ,

$$D \mathbf{G}_{t_0,\varphi}(t) = \text{sgn}(t_0 - t) e^{-[(t_0-t)DB^*]} \chi^{\text{sgn}(t_0-t)}(DB^*) D \varphi. \quad (7.5)$$

The following lemma is a rewording of [15, Lemma 7.4].

**Lemma 7.3.4.** *Let  $F$  solve  $(\text{CR})_{DB}$ . Fix  $\varphi \in L^2(\mathbb{R}^n)$ ,  $t_0 \in \mathbb{R}_+$ , and let  $\eta \in \text{Lip}(\mathbb{R}_+ : \mathbb{R})$  and  $\chi \in \text{Lip}(\mathbb{R}^n : \mathbb{R})$  be compactly supported in  $\mathbb{R}_{+,t_0}$  and  $\mathbb{R}^n$  respectively. Then we have, with absolutely convergent integrals,*

$$\begin{aligned} & \iint_{\mathbb{R}_+^{1+n}} \langle \eta'(t) \chi(x) B^* D \mathbf{G}_{t_0,\varphi}(t, x), F(t, x) \rangle \, dx \, dt \\ &= \iint_{\mathbb{R}_+^{1+n}} \langle \eta(t) B^* [D, m_\chi] \partial_t \mathbf{G}_{t_0,\varphi}(t, x), F(t, x) \rangle \, dx \, dt \end{aligned} \quad (7.6)$$

where  $m_\chi$  denotes the multiplication operator on  $L^2(\mathbb{R}^n)$  with symbol  $\chi$ .

As a corollary, under an integrability condition involving  $F$  and  $\varphi$ , we can obtain the following.

**Corollary 7.3.5.** *Let  $F$ ,  $\varphi$ , and  $t_0$  be as in the statement of Lemma 7.3.4. Suppose also that for all compact  $K \subset \mathbb{R}_{+,t_0}$  we have*

$$\mathbf{1}_K(t) |B^* D\mathbf{G}_{t_0,\varphi}(t, x)| |F(t, x)| \in L^1(\mathbb{R}_+^{1+n}). \quad (7.7)$$

*Then for all  $\eta \in \text{Lip}(\mathbb{R}_+ : \mathbb{R})$  compactly supported in  $\mathbb{R}_{+,t_0}$ , we have the absolutely convergent integral*

$$\iint_{\mathbb{R}_+^{1+n}} \langle \eta'(t) B^* D\mathbf{G}_{t_0,\varphi}(t, x), F(t, x) \rangle dt dx = 0. \quad (7.8)$$

*Proof.* Fix  $\chi \in \text{Lip}(\mathbb{R}^n : \mathbb{R})$  with  $\chi(x) = 1$  for all  $x \in B(0, 1)$ , and for  $R > 0$  define  $\chi_R(x) := \chi(x/R)$ . Then  $\chi_R \rightarrow 1$  and  $[D, m_{\chi_R}] \rightarrow 0$  pointwise as  $R \rightarrow \infty$ ,<sup>4</sup> since  $\|[D, m_{\chi_R}]\|_\infty \lesssim R^{-1} \|\nabla \chi\|_\infty$ . Condition (7.7) applied with  $K = \text{supp } \eta$ , the fact that  $\partial_t \mathbf{G}_{t_0,\varphi} = B^* D\mathbf{G}_{t_0,\varphi}$ , and boundedness of  $\eta$  and  $\eta'$  imply

$$\begin{aligned} |\eta'(t) B^* D\mathbf{G}_{t_0,\varphi}(t, x)| |F(t, x)| &\in L^1(\mathbb{R}_+^{1+n}) \quad \text{and} \\ |\eta(t) \partial_t \mathbf{G}_{t_0,\varphi}(t, x)| |F(t, x)| &\in L^1(\mathbb{R}_+^{1+n}). \end{aligned}$$

This allows us to deduce (7.8) from the equality of Lebesgue integrals (7.6) and dominated convergence.  $\square$

Now, assuming that (7.7) holds, we can conclude the following.

**Corollary 7.3.6.** *Let  $F$ ,  $\varphi$ , and  $t_0$  be as in the statement of Lemma 7.3.4. Assume also that condition (7.7) is satisfied. Then for sufficiently small  $\varepsilon > 0$  we have*

$$\begin{aligned} \int_{t_0+\varepsilon}^{t_0+2\varepsilon} \int_{\mathbb{R}^n} \langle B^* D\mathbf{G}_{t_0,\varphi}(t, x), F(t, x) \rangle dx dt \\ = \int_{t_0+(2\varepsilon)^{-1}}^{t_0+\varepsilon^{-1}} \int_{\mathbb{R}^n} \langle B^* D\mathbf{G}_{t_0,\varphi}(t, x), F(t, x) \rangle dx dt \end{aligned} \quad (7.9)$$

and

$$\begin{aligned} - \int_{\varepsilon}^{2\varepsilon} \int_{\mathbb{R}^n} \langle B^* D\mathbf{G}_{t_0,\varphi}(t_0 - t, x), F(t_0 - t, x) \rangle dx dt \\ = \int_{\varepsilon}^{2\varepsilon} \int_{\mathbb{R}^n} \langle B^* D\mathbf{G}_{t_0,\varphi}(t, x), F(t, x) \rangle dx dt. \end{aligned} \quad (7.10)$$

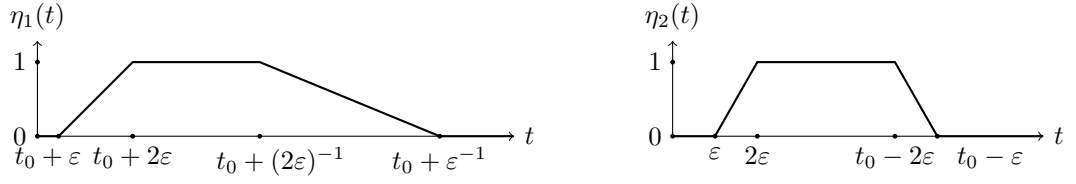
*These are all absolutely convergent integrals.*

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<sup>4</sup>More precisely,  $[D, m_{\chi_R}]$  is given by multiplication with a function that tends to 0 pointwise.



Figure 7.3: The functions  $\eta_1$  and  $\eta_2$ .



*Proof.* As in [15, §8, Step 1b] this follows from applying Corollary 7.3.5 with the piecewise linear functions  $\eta_1, \eta_2 \in \text{Lip}(\mathbb{R}_+ : \mathbb{R})$  drawn in Figure 7.3, where we impose  $\varepsilon < \min(t_0/4, 1/4, 1/t_0)$  (we have carried out a change of variables in the left hand side of (7.10)).  $\square$

### 7.3.3 Proof of Theorem 7.3.1

Recall that part (i) has already been proven in Subsection 7.3.1; here we prove part (ii).

All of the results in this section are valid for  $\mathbf{p} = (p, s)$  such that  $p \leq 2$  and  $s < 0$ . We do not ‘fix’ such a  $\mathbf{p}$ , however, because in the final step we will invoke prior results with a different choice of  $\mathbf{p}$ .

#### Step 1: Verification and application of initial limiting arguments.

**Lemma 7.3.7.** *Let  $\varphi \in L^2(\mathbb{R}^n)$ . Then we have  $\mathbf{1}_{K \times \mathbb{R}^n} B^* D\mathbf{G}_{t_0, \varphi} \in X^{\mathbf{p}'}$  for all compact  $K \subset \mathbb{R}_{+, t_0}$ , with*

$$\|\mathbf{1}_{K \times \mathbb{R}^n} B^* D\mathbf{G}_{t_0, \varphi}\|_{X^{\mathbf{p}'}} \lesssim \|\varphi\|_2 \text{dist}(K, t_0)^{-1} K_-^{s+n\delta_{p,2}} \quad (7.11)$$

where  $K_- = \inf(K)$ .

*Proof.* First we note that the estimate

$$\|\mathbf{1}_{K \times \mathbb{R}^n} B^* D\mathbf{G}_{t_0, \varphi}\|_{X_{-s-n\delta_{p,2}}^2} \lesssim \|\varphi\|_2 \text{dist}(K, t_0)^{-1} K_-^{s+n\delta_{p,2}}$$

can be shown by writing

$$\begin{aligned} \|\mathbf{1}_{K \times \mathbb{R}^n} B^* D\mathbf{G}_{t_0, \varphi}\|_{X_{-s-n\delta_{p,2}}^2} &= \left( \int_{K_-}^{K_+} \left\| t^{s+n\delta_{p,2}} B^* D\mathbf{G}_{t_0, \varphi} \right\|_2^2 \frac{dt}{t} \right)^{1/2} \\ &\lesssim \|\varphi\|_2 \left( \int_{K_-}^{K_+} t^{2(s+n\delta_{p,2})} \text{dist}(K, t_0)^{-2} \frac{dt}{t} \right)^{1/2} \end{aligned} \quad (7.12)$$

$$\lesssim \|\varphi\|_2 \text{dist}(K, t_0)^{-1} K_-^{s+n\delta_{p,2}}. \quad (7.13)$$

The estimate (7.12) follows by writing

$$B^* D \mathbf{G}_{t_0, \varphi} = \frac{\operatorname{sgn}(t_0 - t)}{t_0 - t} (t_0 - t) B^* D e^{-[(t_0 - t) B^* D]} \chi^{\operatorname{sgn}(t_0 - t)} (B^* D) \mathbb{P}_{\overline{\mathcal{R}(B^* D)}} \varphi$$

and noting that the operator

$$(t_0 - t) B^* D e^{-[(t_0 - t) B^* D]} \chi^{\operatorname{sgn}(t_0 - t)} (B^* D) \mathbb{P}_{\overline{\mathcal{R}(B^* D)}}$$

is bounded on  $L^2(\mathbb{R}^n)$  uniformly in  $t \in \mathbb{R}_{+, t_0}$ , and that  $|(t_0 - t)^{-1}| \lesssim \operatorname{dist}(K, t_0)^{-1}$  for  $t \in K$ . Then (7.13) follows because  $s + n\delta_{p,2}$  is negative whenever  $s < 0$  and  $p < 2$ .

Now use the  $X$ -space embeddings to write

$$X_{-s-n\delta_{p,2}}^2 \hookrightarrow X^{\mathbf{p}'},$$

from which follows (7.11).  $\square$

**Corollary 7.3.8.** *Let  $\varphi \in L^2(\mathbb{R}^n)$ , and suppose that  $F \in X^{\mathbf{p}}$  solves  $(\operatorname{CR})_{DB}$ . Then*

$$\lim_{\varepsilon \rightarrow 0} \int_{t_0 + \varepsilon}^{t_0 + 2\varepsilon} \int_{\mathbb{R}^n} \langle B^* D \mathbf{G}_{t_0, \varphi}(t, x), F(t, x) \rangle dx dt = 0. \quad (7.14)$$

*Proof.* For  $\varepsilon > 0$  small the previous lemma yields

$$\left\| \mathbf{1}_{[t_0 + (2\varepsilon)^{-1}, t_0 + \varepsilon^{-1}] \times \mathbb{R}^n} B^* D \mathbf{G}_{t_0, \varphi} \right\|_{X^{\mathbf{p}'}} \lesssim \|\varphi\|_2 (2\varepsilon) (t_0 + (2\varepsilon)^{-1})^{s+n\delta_{p,2}},$$

which decays as  $\varepsilon \rightarrow 0$  since  $s + n\delta_{p,2}$  is negative when  $s < 0$  and  $p \leq 2$ . Therefore in particular, by  $X$ -space duality, condition (7.7) is satisfied, and by boundedness of the above quasinorms as  $\varepsilon \rightarrow 0$  we can take the  $\varepsilon \rightarrow 0$  limit in (7.9) to obtain (7.14).  $\square$

## Step 2: Semigroup property of $F$ .

**Lemma 7.3.9.** *Suppose  $F \in X^{\mathbf{p}}$  solves  $(\operatorname{CR}_{DB})$ . Then  $F \in C^\infty(\mathbb{R}_+ : \mathbb{H}_{DB}^2)$ ,  $F(t) \in \mathcal{D}(DB)$  for all  $t > 0$ , and  $\partial_t F + DBF = 0$  holds strongly in  $C^\infty(\mathbb{R}_+ : \mathbb{H}_{DB}^2)$ .*

*Proof.* We already have that  $F \in C^\infty(\mathbb{R}_+ : L_{\operatorname{loc}}^2(\mathbb{R}^n))$  from Proposition 7.1.1, and furthermore that  $\partial_t F \in X^{\mathbf{p}-1}$ . Hence we have  $F(t_0), (\partial_t F)(t_0) \in E^{\mathbf{p}}$  for all  $t_0 \in \mathbb{R}_+$ , and therefore by the slice space containments of Proposition 5.1.42 we obtain  $F(t_0), (\partial_t F)(t_0) \in L^2$  for all  $t_0 \in \mathbb{R}_+$ . Therefore  $F(t_0) \in \mathcal{D}(DB)$  for all  $t_0 \in \mathbb{R}_+$ , and  $\partial_t F + DBF = 0$  holds in  $L^2$ . We can iterate this argument by

reapplying  $\partial_t$ , as this preserves the property of solving  $(\text{CR})_{DB}$  as well as the previously stated  $L^2$  containments, so we obtain  $F \in C^\infty(\mathbb{R}_+ : L^2)$ .

Now since  $\lim_{t_0 \rightarrow \infty} F(t_0) = 0$  in  $L^2$  (Lemma 7.2.4), we can write

$$F(t_0) = - \int_{t_0}^{\infty} (\partial_t F)(\tau) d\tau = - \int_{t_0}^{\infty} DB(F(\tau)) d\tau \in \overline{\mathcal{R}(DB)}$$

by the fundamental theorem of calculus. Therefore  $F(t_0) \in \mathbb{H}_{DB}^2$  for all  $t_0$ , and since the  $\mathbb{H}_{DB}^2$ -norm is equivalent to the  $L^2$ -norm when restricted to  $\overline{\mathcal{R}(DB)}$ , this completes the proof.  $\square$

**Lemma 7.3.10.** *Suppose that  $F \in X^{\mathbf{p}}$  solves  $(\text{CR})_{DB}$ . Then for all  $t_0 > 0$  and  $\tau \geq 0$  we have  $F(t_0) \in \mathbb{H}_{DB}^{2,+} = \overline{\mathcal{R}(DB)}^+$  and*

$$F(t_0 + \tau) = e^{-\tau DB}(F(t_0)). \quad (7.15)$$

*Proof.* For all  $\varphi \in L^2(\mathbb{R}^n)$ , the function  $t \mapsto B^* D \mathbf{G}_{t_0, \varphi}(t)$  is smooth in  $t \in \mathbb{R}_{+, t_0}$  with values in  $\mathbb{H}_{B^*D}^2$  and with

$$\lim_{t \downarrow t_0} B^* D \mathbf{G}_{t_0, \varphi}(t) = -B^* D \chi^-(B^* D) \mathbb{P}_{\overline{\mathcal{R}(B^* D)}} \varphi$$

in  $\mathbb{H}_{B^*D}^2$ . Furthermore, by Lemma 7.3.9,  $t \mapsto F(t)$  is smooth in  $t \in \mathbb{R}_+$  with values in  $\mathbb{H}_{DB}^2$ . Therefore we may write for all  $\varphi \in L^2(\mathbb{R}^n)$ , using (7.14) from Corollary 7.3.8,

$$\begin{aligned} 0 &= \lim_{\varepsilon \rightarrow 0} \int_{t_0 + \varepsilon}^{t_0 + 2\varepsilon} \int_{\mathbb{R}^n} \langle B^* D \mathbf{G}_{t_0, \varphi}(t, x), F(t, x) \rangle dx dt \\ &= \lim_{\varepsilon \rightarrow 0} \int_{t_0 + \varepsilon}^{t_0 + 2\varepsilon} \langle B^* D \mathbf{G}_{t_0, \varphi}(t), F(t) \rangle_{\mathbb{H}_{B^*D}^2} dt \\ &= -\langle B^* D \chi^-(B^* D) \mathbb{P}_{\overline{\mathcal{R}(B^* D)}} \varphi, F(t_0) \rangle_{\mathbb{H}_{B^*D}^2}. \end{aligned} \quad (7.16)$$

Hence for all  $\phi \in \overline{\mathcal{R}(B^* D)}$  and all  $\delta > 0$ , since  $e^{-\delta[B^* D]}$  maps  $\mathbb{H}_{B^*D}^{2,-}$  into itself, applying (7.16) to  $\varphi = e^{-\delta[B^* D]}\phi$  yields

$$\langle B^* D e^{\delta B^* D} \chi^-(B^* D) \phi, F(t_0) \rangle_{\mathbb{H}_{B^*D}^2} = 0. \quad (7.17)$$

The subspace

$$\{B^* D e^{\delta B^* D} \chi^-(B^* D) \phi : \phi \in \overline{\mathcal{R}(B^* D)}\} \subset L^2(\mathbb{R}^n)$$

is dense in  $\mathbb{H}_{B^*D}^{2,-}$  (see [15, p. 28]), so (7.17) and the decomposition  $\mathbb{H}_{DB}^2 = \mathbb{H}_{DB}^{2,+} \oplus \mathbb{H}_{DB}^{2,-}$  imply that  $F(t_0) \in \mathbb{H}_{DB}^{2,+}$ .

Now we will derive the semigroup equation (7.15). For all  $\delta \geq 0$  and  $\varphi \in \mathbb{H}_{B^*D}^2$ , define

$$\varphi_\delta := e^{-\delta[B^*D]}\varphi$$

and

$$I_{t_0,\varphi}^{\varepsilon,\delta} := \int_{\varepsilon}^{2\varepsilon} \langle B^*D\mathbf{G}_{t_0,\varphi_\delta}(t), F(t) \rangle_{\mathbb{H}_{B^*D}^2} dt.$$

Then by (7.10), using the same argument as before to write everything in terms of  $\mathbb{H}_{B^*D}^2$ -duality, we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} I_{t_0,\varphi}^{\varepsilon,\delta} &= - \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{2\varepsilon} \langle B^*De^{-tB^*D}e^{-\delta B^*D}\chi^+(B^*D)\varphi, F(t_0 - t) \rangle_{\mathbb{H}_{B^*D}^2} dt \\ &= - \langle B^*De^{-\delta B^*D}\chi^+(B^*D)\varphi, F(t_0) \rangle_{\mathbb{H}_{B^*D}^2}. \end{aligned}$$

Therefore for all  $\tau \geq 0$ ,  $\delta \geq 0$ , and  $\varphi \in \mathbb{H}_{B^*D}^2$ , using  $I_{t_0,\varphi}^{\varepsilon,\delta+\tau} = I_{t_0+\delta,\varphi}^{\varepsilon,\tau}$ , we have

$$\begin{aligned} &\langle B^*De^{-\delta B^*D}\chi^+(B^*D)\varphi, e^{-\tau DB}(F(t_0)) \rangle_{\mathbb{H}_{B^*D}^2} \\ &= \langle B^*De^{-(\delta+\tau)B^*D}\chi^+(B^*D)\varphi, F(t_0) \rangle_{\mathbb{H}_{B^*D}^2} \\ &= - \lim_{\varepsilon \rightarrow 0} I_{t_0,\varphi}^{\varepsilon,\delta+\tau} \\ &= - \lim_{\varepsilon \rightarrow 0} I_{t_0+\tau,\varphi}^{\varepsilon,\delta} \\ &= \langle B^*De^{-\delta B^*D}\chi^+(B^*D)\varphi, F(t_0 + \tau) \rangle_{\mathbb{H}_{B^*D}^2}. \end{aligned}$$

As before, the subspace  $\{B^*De^{-\delta B^*D}\chi^+(B^*D)\varphi : \varphi \in \mathbb{H}_{B^*D}^2\}$  is dense in  $\mathbb{H}_{B^*D}^{2,+}$ , so by duality we have  $F(t_0 + \tau) = e^{-\tau DB}F(t_0)$  in  $\mathbb{H}_{DB}^{2,+}$  for all  $t_0 > 0$  and all  $\tau \geq 0$ .  $\square$

### Step 3: Completing the proof.

**Proposition 7.3.11** (Existence of boundary trace). *Suppose that  $F \in X^{\mathbf{p}}$  solves  $(\text{CR})_{DB}$ , and let  $\mathbf{X}_{DB}^{\mathbf{p}}$  be a completion of  $\mathbb{X}_{DB}^{\mathbf{p}}$ . Then there exists a unique  $F_0 \in \mathbf{X}_{DB}^{\mathbf{p},+}$  such that  $F = \mathbf{C}_{DB}^+ F_0$ . Furthermore,  $\|F_0\|_{\mathbf{X}_{DB}^{\mathbf{p}}} \lesssim \|F\|_{X^{\mathbf{p}}}$ .*

*Proof.* Fix an exponent  $\tilde{\mathbf{p}}$  with  $i(\tilde{\mathbf{p}}) \in (1, 2]$  and  $\theta(\tilde{\mathbf{p}}) < 0$  such that  $\mathbf{p} \hookrightarrow \tilde{\mathbf{p}}$  (when  $p > 1$  we may take  $\tilde{\mathbf{p}} = \mathbf{p}$ ). By Lemma 7.3.10 we have  $F(t_0) \in \mathbb{H}_{DB}^{2,+} \cap \mathcal{D}(DB)$  for

all  $t_0 > 0$ . We can then estimate

$$\|F(t_0)\|_{\mathbb{X}_{DB}^{\tilde{\mathbf{p}}}} \simeq \|DBF(t_0)\|_{\mathbb{X}_{DB}^{\tilde{\mathbf{p}}-1}} \quad (7.18)$$

$$= \left\| \tau \mapsto e^{-\tau DB} (DBF)(t_0) \right\|_{X^{\tilde{\mathbf{p}}-1}} \quad (7.19)$$

$$= \left\| \tau \mapsto DBF(t_0 + \tau) \right\|_{X^{\tilde{\mathbf{p}}-1}} \quad (7.20)$$

$$= \|S_{t_0} DBF\|_{X^{\tilde{\mathbf{p}}-1}} \lesssim \|DBF\|_{X^{\tilde{\mathbf{p}}-1}} \quad (7.21)$$

$$= \|\partial_t F\|_{X^{\tilde{\mathbf{p}}-1}} \lesssim \|F\|_{X^{\tilde{\mathbf{p}}}} \quad (7.22)$$

$$\lesssim \|F\|_{X^{\mathbf{p}}} . \quad (7.23)$$

The first line (7.18) is from Corollary 6.1.14. Line (7.19) comes from Theorem 6.1.25. Line (7.20) comes from Lemma 7.3.10, (7.21) comes from Proposition 5.1.36 because  $i(\tilde{\mathbf{p}} - 1) \leq 2$  and  $s(\tilde{\mathbf{p}} - 1) \leq -1/2$ , (7.22) comes from Proposition 7.1.1, and finally (7.23) follows from  $X$ -space embeddings by  $\mathbf{p} \hookrightarrow \tilde{\mathbf{p}}$ . Therefore  $F(t_0) \in \mathbb{X}_{DB}^{\tilde{\mathbf{p}},+}$  uniformly in  $t_0 > 0$ .

Since  $\mathbf{X}_{DB}^{\tilde{\mathbf{p}},+}$  is the dual of  $\mathbf{X}_{B^*D}^{\tilde{\mathbf{p}}',+}$  for any completion  $\mathbf{X}_{B^*D}^{\tilde{\mathbf{p}}',+}$  of  $\mathbb{X}_{B^*D}^{\tilde{\mathbf{p}}',+}$ , there exists a sequence  $t_k \downarrow 0$  and an  $F_0 \in \mathbf{X}_{DB}^{\tilde{\mathbf{p}},+}$  such that  $F(t_k)$  converges weakly to  $F_0$  in  $\mathbf{X}_{DB}^{\tilde{\mathbf{p}},+}$  as  $k \rightarrow \infty$ . We thus have for all  $\varphi \in \mathbb{X}_{B^*D}^{\tilde{\mathbf{p}}',+}$  and for all  $\tau > 0$ ,

$$\begin{aligned} \langle \varphi, e^{-\tau DB} F_0 \rangle_{\mathbf{X}_{B^*D}^{\tilde{\mathbf{p}}',+}} &= \langle e^{-\tau B^*D} \varphi, F_0 \rangle_{\mathbf{X}_{B^*D}^{\tilde{\mathbf{p}}',+}} \\ &= \lim_{k \rightarrow \infty} \langle e^{-\tau B^*D} \varphi, F(t_k) \rangle_{\mathbf{X}_{B^*D}^{\tilde{\mathbf{p}}',+}} \\ &= \lim_{k \rightarrow \infty} \langle \varphi, e^{-\tau DB} F(t_k) \rangle_{\mathbf{X}_{B^*D}^{\tilde{\mathbf{p}}',+}} \\ &= \lim_{k \rightarrow \infty} \langle \varphi, F(t_k + \tau) \rangle_{\mathbf{X}_{B^*D}^{\tilde{\mathbf{p}}',+}} \\ &= \langle \varphi, F(\tau) \rangle_{\mathbf{X}_{B^*D}^{\tilde{\mathbf{p}}',+}} \end{aligned} \quad (7.24)$$

(our notation for duality pairings is explained in Section 4.3), using Lemma 7.3.10 in (7.24). Therefore by density we have  $\mathbf{C}_{DB}^+ F_0 = F$ .

It only remains to show that  $F_0$  is in  $\mathbf{X}_{DB}^{\mathbf{p},+}$ , with the right quasinorm estimate, and uniquely determined. Recall that  $\mathbf{C}_{DB}^+ = \mathbf{Q}_{\text{sgp}, DB}$  when restricted to the positive spectral subspace. Let  $\varphi \in \Psi_\infty^\infty$  be a Calderón sibling of  $\text{sgp}$ . Then  $F_0 = \mathbf{S}_{\varphi, DB} \mathbf{Q}_{\text{sgp}, DB} F_0 = \mathbf{S}_{\varphi, DB} F$ , and so by Proposition 6.1.17 we have  $F_0 \in \mathbf{X}_{DB}^{\mathbf{p}}$  with  $\|F_0\|_{\mathbf{X}_{DB}^{\mathbf{p}}} \lesssim \|F\|_{X^{\mathbf{p}}}$ . In fact, since  $F(t_0) \in \mathbb{H}_{DB}^{2,+}$  for all  $t_0 > 0$ , we find that  $F$  is in the positive subspace  $\mathbf{X}_{DB}^{\mathbf{p},+}$ . Uniqueness follows by injectivity of  $\mathbf{Q}_{\text{sgp}, DB}$  (Proposition 6.1.17).  $\square$

This completes the proof of Theorem 7.3.1.

### 7.3.4 Proof of Theorem 7.3.2

Recall that part (i) has already been proven in Subsection 7.3.1; here we prove part (ii).

Our proof roughly follows that of [15, Theorem 1.3], arguing via a series of rather technical lemmas. In this section we will continually assume that  $\mathbf{p}$  satisfies the assumptions of Theorem 7.3.2. Most of the lemmas work without assuming  $\mathbf{p}^\heartsuit \in I(\mathbf{X}, DB^*)$ , but we gain nothing from dropping this assumption.

#### Step 1: Establishing a good class of test functions.

We define the following class of test functions for  $\mathbb{X}_{DB}^{\mathbf{p}}$ :

$$\mathbb{D}^{\mathbf{p}}(X) := \left\{ \varphi \in \mathcal{D}(D) : D\varphi \in \mathbb{X}_{DB^*}^{\mathbf{p}^\heartsuit}, \chi^\pm(DB^*)D\varphi \in E^{\mathbf{p}^\heartsuit} \right\}.$$

This is large enough to contain the Schwartz functions and to be stable under the action of various operators, yet it is restrictive enough to let us exploit slice space containments.

**Lemma 7.3.12.** *The Schwartz class  $\mathcal{S}(\mathbb{R}^n : \mathbb{C}^{m(1+n)})$  is contained in  $\mathbb{D}^{\mathbf{p}}(X)$ .*

*Proof.* Suppose  $\varphi \in \mathcal{S}$ . Then  $\varphi \in \mathcal{D}(D)$  and  $D\varphi \in \mathbb{X}_D^{\mathbf{p}^\heartsuit} = \mathbb{X}_{DB^*}^{\mathbf{p}^\heartsuit}$  by the assumption on  $\mathbf{p}$ . It remains to show that  $\chi^\pm(DB^*)D\varphi \in E^{\mathbf{p}^\heartsuit}$ , and this takes some work. This is a modification of the argument of [15, Lemma 8.10].

Since  $D\varphi \in \mathcal{S} \subset E^{\mathbf{p}^\heartsuit}$  (Proposition 5.1.43) and since

$$D\varphi = \chi^+(DB^*)D\varphi + \chi^-(DB^*)D\varphi$$

it suffices to show that  $\chi^+(DB^*)D\varphi$  is in  $E^{\mathbf{p}^\heartsuit}$ .

Define  $\psi \in \Psi_N^\infty$ , with  $N$  large to be chosen later, by

$$\psi(z) := \frac{[z]^N e^{-[z]}}{N!}.$$

Then for all  $t \in \mathbb{R} \setminus \{0\}$  we have

$$\int_0^\infty \psi(st) \frac{ds}{s} = \frac{1}{N!} \int_0^\infty s^N e^{-s} \frac{ds}{s} = 1,$$

so by holomorphy we have

$$\int_0^\infty \psi(sz) \frac{ds}{s} = 1$$

for all  $z \in S_\mu$ . By the same argument, along with integration by parts and induction on  $N$ , for all  $z \in S_\mu$  we have

$$\int_1^\infty \psi(sz) \frac{ds}{s} = P([z])e^{-[z]}$$

where  $P$  is a real polynomial of degree  $N - 1$ . Therefore by functional calculus on  $\overline{\mathcal{R}(DB^*)}$  we may write

$$\chi^+(DB^*)D\varphi = \int_0^1 (\psi\chi^+)(sDB^*)D\varphi \frac{ds}{s} + P(DB^*)e^{-DB^*}\chi^+(DB^*)D\varphi.$$

By Lemma 7.1.3 (using  $i(\mathbf{p}^\vee) < 2$  and  $\theta(\mathbf{p}^\vee) < 0$ ) we have

$$P(DB^*)e^{-DB^*}\chi^+(DB^*)D\varphi \in E^{\mathbf{p}^\vee},$$

so it suffices to show that

$$\int_0^1 (\psi\chi^+)(sDB^*)D\varphi \frac{ds}{s} \in E^{\mathbf{p}^\vee}.$$

For  $f \in L^2(\mathbb{R}^n)$  write

$$G(f) := \int_0^1 (\psi\chi^+)(sDB^*)f \frac{ds}{s};$$

since  $\psi\chi^+ \in \Psi_+^\infty$  this is defined for all  $f \in L^2(\mathbb{R}^n)$  (not just  $f \in \overline{\mathcal{R}(DB^*)}$ ). Note that the family  $((\psi\chi^+)(sDB^*))_{s>0}$  satisfies off-diagonal estimates of order  $N$  (Theorem 5.2.8). For  $Q, R \in \mathcal{D}_1$  (recall that  $\mathcal{D}_1$  is the set of standard dyadic cubes in  $\mathbb{R}^n$  with sidelength 1) with  $d(Q, R) \geq 1$  we can estimate

$$\begin{aligned} \|G(\mathbf{1}_R D\varphi)\|_{L^2(Q)} &= \left( \int_Q \left| \int_0^1 (\psi\chi^+)(sDB^*)\mathbf{1}_R D\varphi(x) \frac{ds}{s} \right|^2 dx \right)^{1/2} \\ &\leq \int_0^1 \left\| (\psi\chi^+)(sDB^*)\mathbf{1}_R D\varphi \right\|_{L^2(Q)} \frac{ds}{s} \\ &\lesssim \int_0^1 \left( \frac{d(Q, R)}{s} \right)^{-N} \frac{ds}{s} \|\mathbf{1}_R D\varphi\|_2 \\ &\simeq d(Q, R)^{-N} \|\mathbf{1}_R D\varphi\|_2. \end{aligned}$$

For all other  $Q, R \in \mathcal{D}_1$  we have instead

$$\|G(\mathbf{1}_R D\varphi)\|_{L^2(Q)} \lesssim \|\mathbf{1}_R D\varphi\|_2.$$

Therefore by the discrete characterisation of slice spaces (Proposition 5.1.45), writing  $R \sim Q$  to mean that  $\text{dist}(R, Q) = 0$  and noting that  $\text{dist}(R, Q) \geq 1$  if  $R \not\sim Q$ ,

$$\begin{aligned}
\|G(D\varphi)\|_{E^{\mathbf{p}^\vee}} &\simeq \left( \sum_{Q \in \mathcal{D}_1} \|G(D\varphi)\|_{L^2(Q)}^{i(\mathbf{p}^\vee)} \right)^{1/i(\mathbf{p}^\vee)} \\
&\lesssim \left( \sum_{Q \in \mathcal{D}_1} \left( \sum_{R \sim Q} + \sum_{R \not\sim Q} \right) \|G(\mathbf{1}_R D\varphi)\|_{L^2(Q)}^{i(\mathbf{p}^\vee)} \right)^{1/i(\mathbf{p}^\vee)} \\
&\lesssim \left( \sum_{\substack{Q \in \mathcal{D}_1 \\ R \sim Q}} \|D\varphi\|_{L^2(R)}^{i(\mathbf{p}^\vee)} \right)^{1/i(\mathbf{p}^\vee)} + \\
&\quad \left( \sum_{\substack{Q \in \mathcal{D}_1 \\ R \not\sim Q}} d(Q, R)^{Ni(\mathbf{p}^\vee)} \|D\varphi\|_{L^2(R)}^{i(\mathbf{p}^\vee)} \right)^{1/i(\mathbf{p}^\vee)} \\
&=: \mathbf{I}_1 + \mathbf{I}_2.
\end{aligned}$$

Since the number of cubes  $R \in \mathcal{D}_1$  such that  $R \sim Q$  is uniform in  $Q$ , we have

$$\mathbf{I}_1 \simeq \left( \sum_{R \in \mathcal{D}_1} \|D\varphi\|_{L^2(R)}^{i(\mathbf{p}^\vee)} \right)^{1/i(\mathbf{p}^\vee)} \simeq \|D\varphi\|_{E^{\mathbf{p}^\vee}}.$$

To handle  $\mathbf{I}_2$  write

$$\begin{aligned}
\mathbf{I}_2 &= \left( \sum_{R \in \mathcal{D}_1} \|D\varphi\|_{L^2(R)}^{i(\mathbf{p}^\vee)} \sum_{k=1}^{\infty} k^{Ni(\mathbf{p}^\vee)} |\{Q \in \mathcal{D}_1 : d(Q, R) = k\}| \right)^{1/i(\mathbf{p}^\vee)} \quad (7.25) \\
&\simeq_{N, \mathbf{p}^\vee, n} \|D\varphi\|_{E^{\mathbf{p}^\vee}}
\end{aligned}$$

using that the innermost sum in (7.25) is independent of  $R$  and convergent for  $N$  sufficiently large. Therefore

$$\|G(D\varphi)\|_{E^{\mathbf{p}^\vee}} \lesssim \|D\varphi\|_{E^{\mathbf{p}^\vee}} < \infty$$

which shows that  $\chi^+(DB^*)D\varphi \in E^{\mathbf{p}^\vee}$  and completes the proof.  $\square$

**Lemma 7.3.13.** *We have the following stability properties of  $\mathbb{D}^{\mathbf{p}}(X)$ :*

- (i) *for all  $\delta > 0$  we have  $e^{-\delta[B^*D]}\mathbb{D}^{\mathbf{p}}(X) \subset \mathbb{D}^{\mathbf{p}}(X)$ ,*
- (ii)  *$\chi^\pm(B^*D)\mathbb{D}^{\mathbf{p}}(X) \subset \mathbb{D}^{\mathbf{p}}(X)$ ,*



*Proof.* (i) The function  $[z \mapsto e^{-\delta[z]}]$  is in  $H^\infty$  and has a polynomial limit at 0, so  $e^{-\delta[B^*D]}\varphi$  may be defined for all  $\varphi \in \mathcal{D}(D)$  (not just those in  $\overline{\mathcal{R}(B^*D)}$ ).<sup>5</sup> For all such  $\varphi$  we can write using the similarity of functional calculi

$$D(e^{-\delta[B^*D]}\varphi) = e^{-\delta[DB^*]}D\varphi. \quad (7.26)$$

Since  $D\varphi$  is in  $\mathbb{X}_{DB^*}^{\mathbf{p}^\heartsuit}$ , so is  $D(e^{-\delta[B^*D]}\varphi)$ . To see the slice space containments of spectral projections, write

$$\chi^\pm(DB^*)D(e^{-\delta[B^*D]}\varphi) = e^{-\delta[DB^*]}\chi^\pm(DB^*)D\varphi.$$

By assumption  $\chi^\pm(DB^*)D\varphi$  is in  $E^{\mathbf{p}^\heartsuit} \cap \mathbb{X}_{DB^*}^{\mathbf{p}^\heartsuit, \pm} \subset E^{\mathbf{p}^\heartsuit} \cap \mathbb{X}_{DB^*}^{2, \pm}$ , and by Corollary 6.1.28,  $e^{-\delta[DB^*]}\chi^\pm(DB^*)D\varphi$  is in  $E^{\mathbf{p}^\heartsuit}$ .

(ii) Similarly, we have  $\chi^\pm(B^*D)\mathcal{D}_2(D) \subset \mathcal{D}_2(B^*D)$ , and by similarity of functional calculi

$$D\chi^\pm(B^*D)\varphi_0 = \chi^\pm(DB^*)D\varphi_0 \in \mathbb{X}_{DB^*}^{\mathbf{p}^\heartsuit}$$

and

$$\begin{aligned} \chi^\pm(DB^*)D\chi^\pm(B^*D)\varphi &= \chi^\pm(DB^*)D\varphi_0 \in E^{\mathbf{p}^\heartsuit}, \\ \chi^\mp(DB^*)D\chi^\pm(B^*D)\varphi &= 0 \in E^{\mathbf{p}^\heartsuit}. \end{aligned}$$

□

**Lemma 7.3.14.** *Suppose that  $\varphi \in \mathbb{X}_{B^*D}^{\mathbf{p}'} \cap \mathcal{D}(B^*D)$ . Then  $\chi^\pm(DB^*)e^{-t[DB^*]/2}D\varphi$  is defined and in  $E^{\mathbf{p}'}$  for all  $t > 0$ . Furthermore,  $e^{-[B^*D]/2}\varphi \in \mathbb{D}^{\mathbf{p}}(X)$ .*

*Proof.* Note that  $\mathcal{D}(B^*D) = \mathcal{D}(D)$ . Since  $D\varphi \in \mathbb{X}_{DB^*}^{\mathbf{p}^\heartsuit}$  (Proposition 6.2.6), by Lemma 7.1.3 we find that

$$\chi^\pm(DB^*)e^{-t[DB^*]/2}D\varphi = e^{-t[DB^*]/2}\chi^\pm(DB^*)D\varphi \in E^{\mathbf{p}^\heartsuit} = E^{\mathbf{p}'}. \quad (7.27)$$

To see that  $e^{-[B^*D]/2}\varphi$  is in  $\mathbb{D}^{\mathbf{p}}(X)$ , note that

$$e^{-[B^*D]/2}\varphi \in \mathcal{D}(B^*D) = \mathcal{D}(D),$$

that

$$De^{-[B^*D]/2}\varphi = e^{-[DB^*]/2}D\varphi \in \mathbb{X}_{DB^*}^{\mathbf{p}^\heartsuit},$$

and that

$$\chi^\pm(DB^*)De^{-[B^*D]/2}\varphi = e^{-[DB^*]/2}\chi^\pm(DB^*)D\varphi \in E^{\mathbf{p}^\heartsuit}$$

by (7.27). □

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<sup>5</sup>Although we did not discuss this in Subsection 5.2.1, this is a standard procedure. The representation (7.26) is all we need.

**Step 2: Verification and application of the initial limiting arguments.**

**Lemma 7.3.15.** *Define the operator*

$$\tilde{\mathbf{G}}_{t_0} : \varphi \mapsto \operatorname{sgn}(t_0 - t) e^{-[(t_0 - t)DB^*]} \chi^{\operatorname{sgn}(t_0 - t)} (DB^*) \varphi.$$

Let  $K \subset \mathbb{R}_{+,t_0}$  be compact. Then for all  $k \in \mathbb{N}$ ,  $\mathbf{1}_{K \times \mathbb{R}^n} \tilde{\mathbf{G}}_{t_0}$  is bounded from  $\mathbb{X}_{DB^*}^{\mathbf{p}^\heartsuit}$  to  $X^{\mathbf{p}^\heartsuit+k}$ , and this boundedness is uniform in  $K$  provided  $K_- > t_0 + 1$ .

*Proof.* We will prove the result for tent spaces; the  $Z$ -space result then follows by real interpolation because the assumption on  $\mathbf{p}$  is open in  $(j(\mathbf{p}), \theta(\mathbf{p}))$ .

Suppose  $\varphi \in \mathbb{X}_{DB^*}^{\mathbf{p}^\heartsuit}$  and write  $K = K_0 \cup K_\infty$ , where  $K_0 \subset (0, t_0)$  and  $K_\infty \subset (t_0, \infty)$ . For all  $x \in \mathbb{R}^n$ ,

$$\begin{aligned} & \mathcal{A}^2(\kappa^{\theta(\mathbf{p})+1-k} \mathbf{1}_{K \times \mathbb{R}^n} \tilde{\mathbf{G}}_{t_0}(\varphi))(x)^2 \\ &= \left( \int_{K_0} \int_{B(x,t)} + \int_{K_\infty} \int_{B(x,t)} \right) |t^{\theta(\mathbf{p})+1-k} \tilde{\mathbf{G}}_{t_0}(\varphi)(t, y)|^2 \frac{dy dt}{t^{1+n}} \\ &=: \mathbf{I}_0 + \mathbf{I}_\infty. \end{aligned}$$

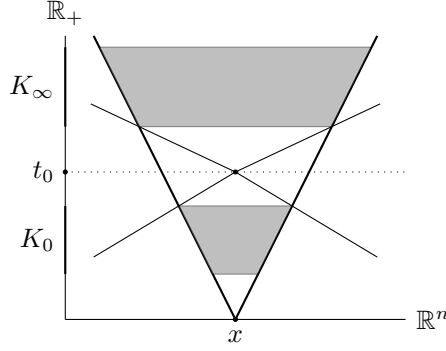
There exists  $\alpha > 0$  (depending on  $K_0$ ) such that if  $(t_0 - \tau, y) \in (K_0 \times \mathbb{R}^n) \cap \Gamma(x)$ , then  $(\tau, y) \in \Gamma^\alpha(x)$  (see Figure 7.4). Thus, using (7.5) and that  $t_0 - \tau \simeq_K \tau$  when  $t_0 - \tau \in K_0$ ,

$$\begin{aligned} \mathbf{I}_0 &\leq \iint_{\Gamma^\alpha(x)} \mathbf{1}_{K_0}(t_0 - \tau) \left| (t_0 - \tau)^{\theta(\mathbf{p})+1-k} \tilde{\mathbf{G}}_{t_0}(\varphi)(t_0 - \tau, y) \right|^2 \frac{dy d\tau}{(t_0 - \tau)^{1+n}} \\ &\lesssim_{K,k} \iint_{\Gamma^\alpha(x)} \left| \tau^{\theta(\mathbf{p})+1} e^{-\tau DB^*} \chi^+(DB^*) \varphi(y) \right|^2 \frac{dy d\tau}{\tau^{1+n}} \end{aligned}$$

Similarly, there exists  $\beta > 0$  such that if  $(t_0 + \sigma, y) \in (K_1 \times \mathbb{R}^n) \cap \Gamma(x)$ , then  $(\sigma, y) \in \Gamma^\beta(x)$ , and using  $2(\theta(\mathbf{p}) + 1) - n - 1 < 0$  we have

$$\begin{aligned} \mathbf{I}_\infty &\leq \iint_{\Gamma^\beta(x)} \mathbf{1}_{K_\infty}(t_0 + \sigma) \left| (t_0 + \sigma)^{\theta(\mathbf{p})+1-k} e^{\sigma DB^*} \chi^-(DB^*) \varphi(y) \right|^2 \frac{dy d\sigma}{(t_0 + \sigma)^{1+n}} \\ &\leq (K_\infty)_-^{-2k} \iint_{\Gamma^\beta(x)} (t_0 + \sigma)^{2(\theta(\mathbf{p})+1)-n-1} \left| e^{\sigma DB^*} \chi^-(DB^*) \varphi(y) \right|^2 dy d\sigma \\ &\leq (K_\infty)_-^{-2k} \iint_{\Gamma^\beta(x)} \sigma^{2(\theta(\mathbf{p})+1)-n-1} \left| e^{\sigma DB^*} \chi^-(DB^*) \varphi(y) \right|^2 dy d\sigma \\ &= (K_\infty)_-^{-2k} \iint_{\Gamma^\beta(x)} \left| \sigma^{\theta(\mathbf{p})+1} e^{\sigma DB^*} \chi^-(DB^*) \varphi(y) \right|^2 \frac{dy d\sigma}{\sigma^{1+n}}. \end{aligned}$$

Figure 7.4: Cones of large aperture, used in Lemma 7.3.15.



Therefore we can estimate

$$\begin{aligned}
& \left\| \mathbf{1}_{K \times \mathbb{R}^n} \tilde{\mathbf{G}}_{t_0}(\varphi) \right\|_{T^{\mathbf{p}^\heartsuit+k}} \\
& \lesssim C(K, k) \left\| e^{-tDB^*} \chi^+(DB^*)\varphi \right\|_{T^{\mathbf{p}^\heartsuit}} + (K_\infty)^{-2k} \left\| e^{tDB^*} \chi^-(DB^*)\varphi \right\|_{T^{\mathbf{p}^\heartsuit}} \\
& \lesssim \|\varphi\|_{\mathbb{H}_{DB^*}^{\mathbf{p}^\heartsuit}} \\
& < \infty,
\end{aligned}$$

using the semigroup characterisation of the  $\mathbb{H}_{DB^*}^{\mathbf{p}^\heartsuit, \pm}$  quasinorm (Theorem 6.1.25), which is valid since  $i(\mathbf{p}^\heartsuit) < 2$  and  $\theta(\mathbf{p}^\heartsuit) < 0$ . Note that if  $K_- > t_0 + 1$  then  $\mathbf{I}_0 = 0$ , and that the aperture  $\beta$  can remain fixed in this argument, which implies the claimed uniformity in  $K$  since  $K_-^{-2k}$  is bounded in  $K_- > t_0 + 1$ .  $\square$

**Corollary 7.3.16.** *Let  $\varphi \in \mathbb{D}^{\mathbf{p}}(X)$  and  $k \in \mathbb{N}$ . Then  $\mathbf{1}_{K \times \mathbb{R}^n} D\mathbf{G}_{t_0, \varphi} \in X^{\mathbf{p}^\heartsuit+k}$  for all compact  $K \subset \mathbb{R}_{+, t_0}$ , with uniform boundedness in  $K$  provided  $K_- > t_0 + 1$ .*

*Proof.* For  $\varphi \in \mathbb{D}^{\mathbf{p}}(X)$  we have  $D\varphi \in \mathbb{X}_{DB^*}^{\mathbf{p}^\heartsuit}$  and  $D\mathbf{G}_{t_0, \varphi} = \tilde{\mathbf{G}}_{t_0}(D\varphi)$ , so this follows from Lemma 7.3.15.  $\square$

For  $k \in \mathbb{N}$ , whenever  $F \in X^{\mathbf{p}-k}$  solves  $(\text{CR})_{DB}$  we can invoke Corollary 7.3.6 when  $\varphi \in \mathbb{D}^{\mathbf{p}}(X)$ , yielding the equalities (7.9) and (7.10) for sufficiently small  $\varepsilon > 0$ .

**Corollary 7.3.17.** *Let  $\varphi \in \mathbb{D}^{\mathbf{p}}(X)$  and  $k \in \mathbb{N}$ , and suppose that  $F \in X^{\mathbf{p}-k}$  solves  $(\text{CR})_{DB}$ . Then*

$$\lim_{\varepsilon \rightarrow 0} \int_{t_0+\varepsilon}^{t_0+2\varepsilon} \int_{\mathbb{R}^n} \langle B^* D\mathbf{G}_{t_0, \varphi}(t, x), F(t, x) \rangle dx dt = 0. \quad (7.28)$$

*Proof.* For  $\varepsilon < 1/2$  we have  $t_0 + (2\varepsilon)^{-1} > t_0 + 1$ , and so by Corollary 7.3.16 we have

$$\mathbf{1}_{[t_0+(2\varepsilon)^{-1}, t_0+\varepsilon^{-1}] \times \mathbb{R}^n} D\mathbf{G}_{t_0, \varphi} \in X^{\mathbf{p}^\heartsuit+k+1}$$

with uniformly bounded quasinorms. Since  $F \in X^{\mathbf{p}-k}$ , and since  $(\mathbf{p} - k)' = \mathbf{p}^\heartsuit + k + 1$ , absolute convergence of the  $X$ -space duality integrals implies that condition (7.7) is satisfied, and also that

$$\int_{t_0+(2\varepsilon)^{-1}}^{t_0+\varepsilon^{-1}} \int_{\mathbb{R}^n} |\langle B^* D\mathbf{G}_{t_0, \varphi}(t, x), F(t, x) \rangle| dx dt \lesssim 1$$

for all  $\varepsilon < 1/2$ . Therefore we can take the limit as  $\varepsilon \rightarrow 0$  of both sides of (7.9) using dominated convergence to conclude that the right hand side vanishes.  $\square$

### Step 3: Weak semigroup properties of solutions.

**Lemma 7.3.18.** *Suppose that  $F \in X^{\mathbf{p}-k}$  solves  $(\text{CR})_{DB}$  for some  $k \in \mathbb{N}$ . When  $t_0 > 0$ ,  $\tau \geq 0$ , and  $\varphi \in \mathbb{D}^{\mathbf{p}}(X)$ , we have*

$$\langle B^* D\varphi, F(t_0 + \tau) \rangle_{E^{\mathbf{p}^\heartsuit}} = \langle B^* e^{-\tau DB^*} \chi^+(DB^*) D\varphi, F(t_0) \rangle_{E^{\mathbf{p}^\heartsuit}}. \quad (7.29)$$

*Proof.* We need to rewrite the integrals in (7.28) and (7.10) in terms of duality of slice spaces. By Proposition 7.1.1,  $F(t)$  is in  $E^{\mathbf{p}}$  for each  $t \in \mathbb{R}_+$ . By Lemma 7.1.3, since  $D\varphi \in \mathbb{X}_{DB^*}^{\mathbf{p}^\heartsuit}$ , we have that  $B^* D\mathbf{G}_{t_0, \varphi}(t)$  is in  $E^{\mathbf{p}^\heartsuit}$ . Hence

$$\int_{\mathbb{R}^n} \langle B^* D\mathbf{G}_{t_0, \varphi}(t, x), F(t, x) \rangle dx = \langle B^* D\mathbf{G}_{t_0, \varphi}(t), F(t) \rangle_{E^{\mathbf{p}^\heartsuit}}$$

by the slice space duality identification of Proposition 5.1.41. Therefore (7.28) and (7.10) can be rewritten as

$$\lim_{\varepsilon \rightarrow 0} \int_{t_0+\varepsilon}^{t_0+2\varepsilon} \langle B^* D\mathbf{G}_{t_0, \varphi}(t), F(t) \rangle_{E^{\mathbf{p}^\heartsuit}} dt = 0 \quad (7.30)$$

and

$$-\lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{2\varepsilon} \langle B^* D\mathbf{G}_{t_0, \varphi}(t_0 - t), F(t_0 - t) \rangle_{E^{\mathbf{p}^\heartsuit}} dt = \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{2\varepsilon} \langle B^* D\mathbf{G}_{t_0, \varphi}(t), F(t) \rangle_{E^{\mathbf{p}^\heartsuit}} dt. \quad (7.31)$$

We need to evaluate these limits by using continuity of the integrands. By Proposition 7.1.1, we have  $F \in C^\infty(\mathbb{R}_+ : E^{\mathbf{p}})$ . By the definition of  $\mathbb{D}^{\mathbf{p}}(X)$ , we have that  $B^* D\mathbf{G}_{t_0, \varphi}(t) \in E^{\mathbf{p}^\heartsuit}$  for all  $t \in \mathbb{R}_{+, t_0}$ , with

$$\begin{aligned} \lim_{t \searrow t_0} B^* D\mathbf{G}_{t_0, \varphi}(t) &= -B^* D\chi^-(B^* D)\mathbb{P}_{\overline{\mathcal{R}(B^* D)}}\varphi \\ &= -B^* \chi^-(DB^*) D\varphi \end{aligned}$$

in  $E^{\mathbf{P}^\heartsuit}$  by Corollary 6.1.28. Therefore (7.30) becomes

$$\langle B^* \chi^-(DB^*) D\varphi, F(t_0) \rangle_{E^{\mathbf{P}^\heartsuit}} = 0. \quad (7.32)$$

Next, we will prove

$$\langle B^* e^{-\tau DB^*} \chi^+(DB^*) D\varphi, F(t_0) \rangle_{E^{\mathbf{P}^\heartsuit}} = \langle B^* \chi^+(DB^*) D\varphi, F(t_0 + \tau) \rangle_{E^{\mathbf{P}^\heartsuit}}. \quad (7.33)$$

by taking the limit of the left hand side of (7.31) and exploiting an algebraic property of the right hand side. Summing (7.32) (at  $t_0 + \tau$ ) and (7.33) will yield (7.29) and complete the proof.

For  $\varphi \in \mathbb{D}^{\mathbf{P}}(X)$  and  $\delta \geq 0$ , define

$$I_{t_0, \varphi}^{\varepsilon, \delta} := \int_{\varepsilon}^{2\varepsilon} \langle B^* D\mathbf{G}_{t_0, \varphi_\delta}(t), F(t) \rangle_{E^{\mathbf{P}^\heartsuit}} dt$$

where  $\varphi_\delta := e^{-\delta[B^*D]}\varphi$ . By Lemma 7.3.13,  $\varphi_\delta$  is in  $\mathbb{D}^{\mathbf{P}}(X)$ , and so we can apply (7.31) to get

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} I_{t_0, \varphi}^{\varepsilon, \delta} &= - \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{2\varepsilon} \langle B^* D\mathbf{G}_{t_0, \varphi_\delta}(t_0 - t), F(t_0 - t) \rangle_{E^{\mathbf{P}^\heartsuit}} dt \\ &= - \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{2\varepsilon} \langle B^* e^{-tDB^*} e^{-\delta DB^*} \chi^+(DB^*) D\varphi(t), F(t) \rangle_{E^{\mathbf{P}^\heartsuit}} dt \\ &= - \langle B^* e^{-\delta DB^*} \chi^+(DB^*) D\varphi(t_0), F(t_0) \rangle_{E^{\mathbf{P}^\heartsuit}} \end{aligned}$$

using the same argument as in the previous paragraph to establish the final equality. A simple computation shows that we have

$$I_{t_0, \varphi}^{\varepsilon, \delta} = I_{t_0 + \delta, \varphi}^{\varepsilon, 0},$$

and so we can conclude

$$\begin{aligned} \langle B^* e^{-\tau DB^*} \chi^+(DB^*) D\varphi, F(t_0) \rangle_{E^{\mathbf{P}^\heartsuit}} &= - \lim_{\varepsilon \rightarrow 0} I_{t_0, \varphi}^{\varepsilon, \tau} \\ &= - \lim_{\varepsilon \rightarrow 0} I_{t_0 + \tau, \varphi}^{\varepsilon, 0} \\ &= \langle B^* \chi^+(DB^*) D\varphi, F(t_0 + \tau) \rangle_{E^{\mathbf{P}^\heartsuit}}, \end{aligned}$$

completing the proof.  $\square$

We can use this lemma, using that  $E^{\mathbf{P}} \subset \mathcal{S}'$ , to see what happens when we test against Schwartz functions.

**Corollary 7.3.19.** *Let  $F$ ,  $t_0$ , and  $\tau$  be as in Lemma 7.3.18, and suppose  $\varphi \in \mathcal{S}$ . Then*

$$-\langle \varphi, (\partial_t F)(t_0 + \tau) \rangle_{\mathcal{S}} = \langle B^* e^{-\tau DB^*} \chi^+(DB^*) D\varphi, F(t_0) \rangle_{E^{\mathbf{P}^\vee}}. \quad (7.34)$$

*Proof.* By Lemma 7.3.12,  $\mathcal{S} \subset \mathbb{D}^{\mathbf{P}}(X)$ , so we can apply Lemma 7.3.18 to  $\varphi$ . Since  $F(t_0 + \tau)$  and  $(\partial_t F)(t_0 + \tau)$  are in  $E^{\mathbf{P}}$ , and  $\varphi$  and  $B^* D\varphi$  are in  $E^{\mathbf{P}^\vee}$ , we can apply integration by parts in slice spaces (Proposition 5.1.44) to derive (7.34).  $\square$

**Step 4: A reproducing formula for  $(\partial_t F)(t_0)$  in terms of higher derivatives.**

**Lemma 7.3.20.** *Let  $t_0 > 0$ ,  $k \in \mathbb{N}_+$ , and suppose that  $F \in X^{\mathbf{P}}$  solves  $(\text{CR})_{DB}$ . Then  $(\partial_t^k F)(t_0) \in \mathbf{X}_D^{\mathbf{P}}$ , with*

$$\left\| (\partial_t^k F)(t_0) \right\|_{\mathbf{X}_D^{\mathbf{P}}} \lesssim t_0^{-k} \|F\|_{X^{\mathbf{P}}}. \quad (7.35)$$

*Proof.* Suppose  $\varphi \in \mathcal{S}$ . First note that since  $\partial_t^{k-1} F$  solves  $(\text{CR})_{DB}$ , and is in  $X^{\mathbf{P}-(k-1)}$  by Proposition 7.1.1, Corollary 7.3.19 yields

$$-\langle \varphi, (\partial_t^k F)(t_0/2 + \tau) \rangle_{\mathcal{S}} = \langle B^* e^{-\tau DB^*} \chi^+(DB^*) D\varphi, (\partial_t^{k-1} F)(t_0/2) \rangle_{E^{\mathbf{P}^\vee}}. \quad (7.36)$$

Applying this with  $\tau = t_0/2$  and using the slice space estimates of Lemma 7.1.3 and Proposition 7.1.1, slice space duality, and  $\mathbf{p}^\vee \in I(\mathbf{X}, DB^*)$ , we have

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} \langle \varphi(x), \partial_t^k F(t_0)(x) \rangle dx \right| \\ & \leq \left\| B^* e^{-t_0 DB^*/2} \chi^+(DB^*) D\varphi \right\|_{E^{\mathbf{P}^\vee}(t_0/2)} \left\| (\partial_t^{k-1} F)(t_0/2) \right\|_{E^{\mathbf{P}+1}(t_0/2)} \\ & \lesssim \|D\varphi\|_{\mathbf{X}_{DB^*}^{\mathbf{P}^\vee}} t_0^{-k} \left\| (\partial_t^{k-1} F)(t_0/2) \right\|_{E^{\mathbf{P}-(k-1)}(t_0)} \\ & \lesssim \|D\varphi\|_{\mathbf{X}_D^{\mathbf{P}^\vee}} t_0^{-k} \left\| \partial_t^{k-1} F \right\|_{X^{\mathbf{P}-(k-1)}} \\ & \lesssim \|\varphi\|_{\mathbf{X}^{\mathbf{P}'}} t_0^{-k} \|F\|_{X^{\mathbf{P}}}. \end{aligned}$$

Since  $\varphi$  was arbitrary, this implies that  $(\partial_t^k F)(t_0) \in (\mathbf{X}^{\mathbf{P}'})' = \mathbf{X}^{\mathbf{P}}$  with the norm estimate (7.35). Furthermore, since  $\partial_t^k F$  solves  $(\text{CR})_{DB}$ , each  $(\partial_t^k F)(t_0)$  is in  $\overline{\mathcal{R}(DB)} = \overline{\mathcal{R}(D)}$ , which implies membership in  $\mathbf{X}_D^{\mathbf{P}}$ .  $\square$

We recall the following elementary lemma (see [15, Lemma 9.2]).

**Lemma 7.3.21.** *Suppose  $k \in \mathbb{N}$  and  $g \in C^k(\mathbb{R}^+ : \mathbb{C})$ , with  $t^j g^{(j)}(t) \rightarrow 0$  as  $t \rightarrow \infty$  for all integers  $0 \leq j \leq k-1$ . Then for all  $t > 0$  we have*

$$g(t) = \frac{(-1)^k}{(k-1)!} \int_t^\infty g^{(k)}(\tau) (\tau - t)^{k-1} d\tau.$$

**Corollary 7.3.22.** *Suppose that  $F \in X^{\mathbf{P}}$  solves  $(\text{CR})_{DB}$ . Then for all  $t_0 > 0$  and  $\varphi \in \mathcal{S}$  we have*

$$\langle \varphi, (\partial_t F)(t_0) \rangle_{\mathcal{S}} = \frac{(-1)^k}{(k-1)!} \int_{t_0}^{\infty} \langle \varphi, (\partial_t^{k+1} F)(t) \rangle_{E^{\mathbf{P}'}} (t - t_0)^{k-1} dt.$$

*Proof.* By Lemma 7.3.20 we have that the function  $t_0 \mapsto (\partial_t F)(t_0)$  is in  $C^\infty(\mathbb{R}_+ : \mathbf{X}_D^{\mathbf{P}})$ . Therefore for all  $\varphi \in \mathcal{S}$  the function  $g_\varphi$  defined by

$$g_\varphi(t_0) := \langle \varphi, (\partial_t F)(t_0) \rangle_{\mathcal{S}}$$

is in  $C^\infty(\mathbb{R}_+ : \mathbb{C})$ , and for  $k \in \mathbb{N}_+$  we have

$$g_\varphi^{(k)}(t_0) = \langle \varphi, (\partial_t^{k+1} F)(t_0) \rangle_{\mathcal{S}} = \langle \varphi, (\partial_t^{k+1} F)(t_0) \rangle_{E^{\mathbf{P}'}}.$$

Furthermore, by the same lemma, we have

$$\begin{aligned} |t_0^k g_\varphi(t_0)| &= t_0^k \left| \langle \varphi, (\partial_t F)(t_0) \rangle_{\mathbf{X}_D^{\mathbf{P}'}} \right| \\ &\lesssim_{\varphi, F} t_0^{-1}, \end{aligned}$$

so the hypotheses of Lemma 7.3.21 are satisfied, and the result follows.  $\square$

#### Step 5: Construction of associated ‘nice’ solutions.

In this step of the proof, given a solution  $F \in X^{\mathbf{P}}$  of  $(\text{CR})_{DB}$ , we will construct distributions modulo polynomials  $\tilde{F}(t_0) \in \mathbf{X}_D^{\mathbf{P}}$  which satisfy the properties we want to show for  $F(t_0)$ . In the remaining steps we will show that  $\tilde{F}(t_0) = F(t_0)$ , which will complete the proof.

**Lemma 7.3.23.** *Suppose  $F \in X^{\mathbf{P}}$  solves  $(\text{CR})_{DB}$ . Then for all  $t_0 \in [0, \infty)$  and for sufficiently large  $N \in \mathbb{N}$  we have*

$$\left\| (t, y) \mapsto t^N (\partial_t^N F)(t_0 + t, y) \right\|_{X^{\mathbf{P}}} \lesssim \|F\|_{X^{\mathbf{P}}}.$$

*Proof.* This is an immediate corollary of Propositions 5.1.36 and 7.1.1.  $\square$

Let  $F \in X^{\mathbf{P}}$  solve  $(\text{CR})_{DB}$ . For  $N \in \mathbb{N}$  large enough that Lemma 7.3.23 applies, define  $\zeta \in \Psi_1^\infty$  by

$$\zeta(z) := c_N z e^{-|z|/2}$$

where  $c_N = (-1)^{N+1}/N!$ . For  $k \in \mathbb{N}$  define  $\chi_k := \mathbf{1}_{[k^{-1}, k] \times B(0, k)}$ , and for all  $t_0 \geq 0$  define

$$\tilde{F}_k(t_0) := \mathbb{S}_{\zeta, DB} \left[ t \mapsto \chi_k t^N (\partial_t^N F) \left( t_0 + \frac{t}{2} \right) \right]. \quad (7.37)$$

By Lemma 7.3.23 we have  $[t \mapsto \chi_k t^N \partial_t^N F(t_0 + \frac{t}{2})] \in X^{\mathbf{p}} \cap X^2$ , and so  $\tilde{F}_k(t_0)$  is a well-defined element of  $\mathbb{X}_{DB}^{\mathbf{p}}$ . Furthermore, since  $\zeta \in \Psi_+^0$ , Proposition 6.1.6 and Lemma 7.3.23 tell us that

$$\begin{aligned} \|\tilde{F}_k(t_0)\|_{\mathbb{X}_{DB}^{\mathbf{p}}} &\lesssim \left\| t \mapsto \chi_k t^N (\partial_t^N F) \left( t_0 + \frac{t}{2} \right) \right\|_{X^{\mathbf{p}}} \\ &\lesssim \|F\|_{X^{\mathbf{p}}}. \end{aligned}$$

Since the functions  $[t \mapsto \chi_k t^N (\partial_t^N F) (t_0 + \frac{t}{2})]$  converge to  $[t \mapsto t^N (\partial_t^N F) (t_0 + \frac{t}{2})]$  in  $X^{\mathbf{p}}$  as  $k \rightarrow \infty$ , given a completion<sup>6</sup>  $\mathbf{X}_{DB}^{\mathbf{p}}$  of  $\mathbb{X}_{DB}^{\mathbf{p}}$ , we get an element  $\tilde{F}(t_0) \in \mathbf{X}_{DB}^{\mathbf{p}}$  defined by

$$\tilde{F}_{t_0} := \mathbf{S}_{\zeta, DB} \left[ t \mapsto t^N (\partial_t^N F) \left( t_0 + \frac{t}{2} \right) \right]$$

and satisfying

$$\|\tilde{F}(t_0)\|_{\mathbf{X}_{DB}^{\mathbf{p}}} \lesssim \|F\|_{X^{\mathbf{p}}}. \quad (7.38)$$

Since  $\mathbf{p}^\heartsuit \in I(\mathbf{X}, DB^*)$ , we can identify  $\mathbf{X}_D^{\mathbf{p}}$  as a completion of  $\mathbb{X}_{DB}^{\mathbf{p}}$ , and so in this case each  $\tilde{F}(t_0) \in \mathbf{X}_D^{\mathbf{p}}$  is a distribution modulo polynomials.

**Lemma 7.3.24.** *Let  $t_0 \geq 0$ . Suppose  $F \in X^{\mathbf{p}}$  solves  $(\text{CR})_{DB}$ , let  $\mathbf{X}_{DB}^{\mathbf{p}}$  be a completion of  $\mathbb{X}_{DB}^{\mathbf{p}}$ , and define  $\tilde{F}(t_0) \in \mathbf{X}_{DB}^{\mathbf{p}}$  as in the previous paragraphs. Suppose also that  $\phi \in \mathbb{X}_{B^*D}^{\mathbf{p}'} \cap \mathcal{D}(B^*D)$ . Then we have*

$$\langle \phi, \tilde{F}(t_0) \rangle_{\mathbf{X}_{B^*D}^{\mathbf{p}'}} = -c_N \int_0^\infty \left\langle t B^* e^{-\frac{t}{2}[DB^*]} D\phi, t^N (\partial_t^N F) \left( t_0 + \frac{t}{2} \right) \right\rangle_{E^{\mathbf{p}'}} \frac{dt}{t} \quad (7.39)$$

*Proof.* First we show the the  $E^{\mathbf{p}'}$  duality pairing (7.39) makes sense. Since  $\phi \in \mathbb{X}_{B^*D}^{\mathbf{p}'} \cap \mathcal{D}(B^*D)$ , Lemma 7.3.14 yields  $e^{-t[DB^*]/2} D\phi \in E^{\mathbf{p}'}$ . Since each  $tB^*$  is a bounded operator on  $E^{\mathbf{p}'}$  (not uniformly in  $t$  of course) we have  $tB^* e^{-t[DB^*]/2} D\phi \in E^{\mathbf{p}'}$ . On the other hand, since  $t \mapsto (\partial_t^N F)(t_0 + t/2)$  solves  $(\text{CR})_{DB}$ , by Proposition 7.1.1 and Lemma 7.3.23 we have

$$\begin{aligned} \|(\partial_t^N F)(t_0 + t/2)\|_{E^{\mathbf{p}-N}(t)} &\lesssim \|t \mapsto (\partial_t^N F)(t_0 + t/2)\|_{X^{\mathbf{p}-N}} \\ &\lesssim \|F\|_{X^{\mathbf{p}}} \end{aligned}$$

for all  $t > 0$ . Therefore the slice space dual pairing in (7.39) is meaningful.

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<sup>6</sup>Recall that when  $\mathbf{p}$  is infinite we always use weak-star completions instead of ordinary completions.



Now write

$$\begin{aligned}
\langle \phi, \tilde{F}(t_0) \rangle_{\mathbf{X}_{B^*D}^{\mathbf{P}'}} &= \lim_{k \rightarrow \infty} \langle \phi, \tilde{F}_k(t_0) \rangle_{\mathbb{X}_{B^*D}^{\mathbf{P}'}} \\
&= \lim_{k \rightarrow \infty} \langle \mathbb{Q}_{\tilde{\zeta}, B^*D} \phi, [t \mapsto \chi_k t^N (\partial_t^N F)(t_0 + t/2)] \rangle_{X^{\mathbf{P}}} \\
&= \langle \mathbb{Q}_{\tilde{\zeta}, B^*D} \phi, [t \mapsto t^N (\partial_t^N F)(t_0 + t/2)] \rangle_{X^{\mathbf{P}}} \\
&= -c_N \iint_{\mathbb{R}_+^{1+n}} \left( t(B^*D e^{-t[B^*D]/2} \phi)(x), t^N (\partial_t^N F)(t_0 + t/2, x) \right) dx \frac{dt}{t} \\
&= -c_N \int_0^\infty \langle t B^* e^{-t[DB^*]/2} D\phi, t^N (\partial_t^N F)(t_0 + t/2) \rangle_{E^{\mathbf{P}'}} \frac{dt}{t}
\end{aligned}$$

using  $\tilde{\zeta} = \zeta$  and the slice space containments from the previous paragraph.  $\square$

Now we will show that the distributions (modulo polynomials)  $(\tilde{F}(t_0))_{t_0 \geq 0}$  are in fact given by the Cauchy operator applied to  $\tilde{F}(0)$ .

**Proposition 7.3.25.** *Let  $F \in X^{\mathbf{P}}$  solve  $(\text{CR})_{DB}$ , fix a completion  $\mathbf{X}_{DB}^{\mathbf{P}}$  of  $\mathbb{X}_{DB}^{\mathbf{P}}$ , and define  $\tilde{F}$  as above. Then for all  $t_0 \geq 0$  we have*

$$\tilde{F}(t_0) = e^{-t_0[DB]} \chi^+(DB) \tilde{F}(0),$$

In particular,  $\tilde{F}(0) \in \mathbf{X}_{DB}^{\mathbf{P},+}$ , and so  $\tilde{F} = \mathbf{C}_{DB}^+(\tilde{F}(0))$ .

*Proof.* Since  $\tilde{F}(t_0) \in \mathbf{X}_{DB}^{\mathbf{P}}$  and since  $\mathbb{X}_{B^*D}^{\mathbf{P}'} \cap \mathcal{D}(B^*D)$  is dense in  $\mathbf{X}_{B^*D}^{\mathbf{P}'}$  (Corollary 6.1.7 and density of  $\mathcal{D}(B^*D)$  in  $\mathbb{X}_{B^*D}^2$ ), it suffices to test against  $\phi \in \mathbb{X}_{B^*D}^{\mathbf{P}'} \cap \mathcal{D}(B^*D)$ . For all such  $\phi$  write

$$\begin{aligned}
&\langle \phi, e^{-t_0[DB]} \chi^+(DB) \tilde{F}(0) \rangle_{\mathbf{X}_{B^*D}^{\mathbf{P}'}} \\
&= \langle e^{-t_0[B^*D]} \chi^+(B^*D) \phi, \tilde{F}(0) \rangle_{\mathbf{X}_{B^*D}^{\mathbf{P}'}} \\
&= -c_N \int_0^\infty \left\langle t B^* e^{-\frac{t}{2}[DB^*]} D \left( e^{-t_0[B^*D]} \chi^+(B^*D) \phi \right), t^N (\partial_t^N F)(t/2) \right\rangle_{E^{\mathbf{P}'}} \frac{dt}{t} \quad (7.40)
\end{aligned}$$

$$\begin{aligned}
&= -c_N \int_0^\infty \left\langle t B^* e^{-t_0[DB^*]} \chi^+(DB^*) D \left( e^{-\frac{t}{2}[B^*D]} \phi \right), t^N (\partial_t^N F)(t/2) \right\rangle_{E^{\mathbf{P}'}} \frac{dt}{t} \\
&= -c_N \int_0^\infty \left\langle t B^* D \left( e^{-\frac{t}{2}[B^*D]} \phi \right), t^N (\partial_t^N F) \left( t_0 + \frac{t}{2} \right) \right\rangle_{E^{\mathbf{P}'}} \frac{dt}{t} \quad (7.41)
\end{aligned}$$

$$\begin{aligned}
&= -c_N \int_0^\infty \left\langle t B^* e^{-\frac{t}{2}[DB^*]} D\phi, t^N (\partial_t^N F) \left( t_0 + \frac{t}{2} \right) \right\rangle_{E^{\mathbf{P}'}} \frac{dt}{t} \\
&= \langle \phi, \tilde{F}(t_0) \rangle_{\mathbf{X}_{B^*D}^{\mathbf{P}'}} \quad (7.42)
\end{aligned}$$

In (7.40) we used that  $e^{-t_0[B^*D]} \chi^+(B^*D)$  maps  $\mathbb{X}_{B^*D}^{\mathbf{P}'} \cap \mathcal{D}(B^*D)$  into itself, and the representation (7.39). In (7.41) we used Lemma 7.3.18, which is valid since  $e^{-t[B^*D]/2} \phi \in \mathbb{D}^{\mathbf{P}}(X)$  (Lemma 7.3.14) and since  $[t \mapsto (\partial_t^N F)(t/2)] \in X^{\mathbf{P}-1}$  solves  $(\text{CR})_{DB}$ . We use the representation (7.39) once more in the last line.  $\square$

This immediately implies the following corollary.

**Corollary 7.3.26.** *Let  $F \in X^{\mathbf{p}}$  solve  $(\text{CR})_{DB}$ . Then  $\tilde{F}(0) \in \mathbf{X}_D^{\mathbf{p},+}$ ,  $\tilde{F}$  is equal to the Cauchy extension  $\mathbf{C}_{DB}^+(\tilde{F}(0))$ , and  $\tilde{F} \in X^{\mathbf{p}}$ .*

*Proof.* All we need to show is that  $\tilde{F}$  is in  $X^{\mathbf{p}}$ . This follows from Theorem 6.2.12.  $\square$

### Step 6: Equality of $\partial_t F$ and $\partial_t \tilde{F}$ .

By Corollary 7.3.26 and Proposition 6.1.24, for  $F \in X^{\mathbf{p}}$  which solves  $(\text{CR})_{DB}$ , the function  $t_0 \mapsto \tilde{F}(t_0)$  is in  $C^\infty(\mathbb{R}_+ : \mathbf{X}_D^{\mathbf{p}})$ . Therefore we can consider  $(\partial_t \tilde{F})(t_0) \in \mathbf{X}_D^{\mathbf{p}}$  as a distribution modulo polynomials.

**Lemma 7.3.27.** *Let  $F \in X^{\mathbf{p}}$  solve  $(\text{CR})_{DB}$ . Then for all  $t_0 > 0$  we have  $(\partial_t F)(t_0) = (\partial_t \tilde{F})(t_0)$  in  $\mathcal{Z}'(\mathbb{R}^n)$ .*

*Proof.* Fix  $\varphi \in \mathcal{Z}$ .<sup>7</sup> For all  $k \in \mathbb{N}$  we have already computed (using that everything is in  $L^2$ )

$$\begin{aligned} & \langle \varphi, \tilde{F}_k(t_0) \rangle_{\mathcal{Z}} \\ &= -c_N \iint_{\mathbb{R}_+^{1+n}} \left( t(B^* e^{-t[DB^*]/2} D\varphi)(x), \chi_k t^N (\partial_t^N F)(t_0 + t/2, x) \right) dx \frac{dt}{t}. \end{aligned} \quad (7.43)$$

Since  $\varphi \in \mathcal{Z}$  we have  $D\varphi \in \mathbf{X}_D^{\mathbf{p}'}$ , so for each  $t > 0$  we may apply the (extended operator)  $e^{-t[DB^*]/2}$  to  $D\varphi$ . We then have

$$\begin{aligned} & \left\| t \mapsto t B^* e^{-t[DB^*]/2} D\varphi \right\|_{X^{\mathbf{p}'}} \\ &= \left\| t \mapsto B^* e^{-t[DB^*]/2} D\varphi \right\|_{X^{\mathbf{p}'\heartsuit}} \\ &\lesssim \|D\varphi\|_{\mathbf{X}_D^{\mathbf{p}'\heartsuit}} \\ &\simeq \|\varphi\|_{\mathbf{X}^{\mathbf{p}'}} < \infty, \end{aligned} \quad (7.44)$$

where (7.44) follows from Proposition 6.1.3 since  $[z \mapsto z e^{-[z]/2}] \in \Psi(\mathbb{X}_D^{\mathbf{p}'\heartsuit})$  (here we use  $i(\mathbf{p}'\heartsuit) < 2$  and  $\theta(\mathbf{p}'\heartsuit) < 0$ ). Since  $[t \mapsto t^N \partial_t^N F(t_0 + t/2)] \in X^{\mathbf{p}}$  (Lemma 7.3.23), the integral (7.43) is uniformly bounded in  $k$  and so we can take the limit

$$\begin{aligned} \langle \varphi, \tilde{F}(t_0) \rangle_{\mathcal{Z}} &= \lim_{k \rightarrow \infty} \langle \varphi, \tilde{F}_k(t_0) \rangle_{\mathcal{Z}} \\ &= -c_N \iint_{\mathbb{R}_+^{1+n}} \left( t(B^* e^{-t[DB^*]/2} D\varphi)(x), t^N (\partial_t^N F)(t_0 + t/2, x) \right) dx \frac{dt}{t} \end{aligned}$$

---

<sup>7</sup>Recall that  $\mathcal{Z}(\mathbb{R}^n)$  is the space of Schwartz functions  $f$  with  $D^\alpha f(0) = 0$  for every multi-index  $\alpha$ .

by dominated convergence. Using dominated convergence again, we can take the derivative:

$$\begin{aligned}
& \langle \varphi, (\partial_t \tilde{F})(t_0) \rangle_{\mathcal{Z}} \\
&= \partial_t \langle \varphi, \tilde{F}(t_0) \rangle_{\mathcal{Z}} \\
&= -c_N \iint_{\mathbb{R}_+^{1+n}} \left( t(B^* e^{-t[DB^*]/2} D\varphi)(x), t^N (\partial_t^{N+1} F)(t_0 + t/2, x) \right) dx \frac{dt}{t} \\
&= -c_N \int_0^\infty \left\langle tB^* e^{-t[DB^*]/2} D\varphi, t^N (\partial_t^{N+1} F)(t_0 + t/2) \right\rangle_{E\mathbb{P}'} \frac{dt}{t}
\end{aligned}$$

using that  $\varphi \in \mathbb{D}^{\mathbb{P}}(X)$  (Lemma 7.3.12) to conclude that the slice space duality pairing is meaningful as in the proof of Lemma 7.3.24.

Now we rearrange:

$$\begin{aligned}
& \left\langle tB^* e^{-t[DB^*]/2} D\varphi, t^N (\partial_t^{N+1} F)(t_0 + t/2) \right\rangle_{E\mathbb{P}'} \\
&= \left\langle tB^* D \left( e^{-t[B^*D]/2} \varphi \right), t^N (\partial_t^{N+1} F)(t_0 + t/2) \right\rangle_{E\mathbb{P}'} \tag{7.45}
\end{aligned}$$

$$= \left\langle tB^* \chi^+(DB^*) D e^{-t[B^*D]/2} \varphi, t^N (\partial_t^{N+1} F)(t_0 + t/2) \right\rangle_{E\mathbb{P}'} \tag{7.46}$$

$$= t^{N+1} \left\langle B^* e^{-t[DB^*]/2} \chi^+(DB^*) D\varphi, (\partial_t^{N+1} F)(t_0 + t/2) \right\rangle_{E\mathbb{P}'} \tag{7.47}$$

$$= t^{N+1} \left\langle B^* D\varphi, (\partial_t^{N+1} F)(t_0 + t) \right\rangle_{E\mathbb{P}'} \tag{7.48}$$

$$= -t^{N+1} \left\langle \varphi, (\partial_t^{N+2} F)(t_0 + t) \right\rangle_{E\mathbb{P}'} . \tag{7.49}$$

The first line (7.45) uses that  $\varphi \in \mathcal{D}(D) = \mathcal{D}(B^*D)$ , (7.46) uses (7.32) and the fact that  $e^{-t[B^*D]/2} \varphi$  is in  $\mathcal{D}^{\mathbb{P}}(X)$  (Lemma 7.3.13), (7.47) is just similarity of functional calculi and rearrangement, (7.48) uses the weak semigroup property (7.29), and (7.49) finishes with integration by parts in slice spaces (Proposition 5.1.44) and  $(\text{CR})_{DB}$ .

Therefore we have

$$\begin{aligned}
\langle \varphi, (\partial_t \tilde{F})(t_0) \rangle_{\mathcal{Z}} &= c_N \int_0^\infty \left\langle \varphi, (\partial_t^{N+2} F)(t_0 + t) \right\rangle_{E\mathbb{P}'} t^{N+1} \frac{dt}{t} \\
&= \frac{(-1)^{N+1}}{N!} \int_{t_0}^\infty \left\langle \varphi, (\partial_t^{N+2} F)(t) \right\rangle_{E\mathbb{P}'} (t - t_0)^N dt.
\end{aligned}$$

Finally, applying Corollary 7.3.22 with  $k = N + 1$ , we get

$$\langle \varphi, (\partial_t F)(t_0) \rangle_{\mathcal{Z}} = \langle \varphi, (\partial_t \tilde{F})(t_0) \rangle_{\mathcal{Z}}$$

for all  $\varphi \in \mathcal{Z}$  and all  $t_0 > 0$ . Therefore we have  $(\partial_t F)(t_0) = (\partial_t \tilde{F})(t_0)$  in  $\mathcal{Z}'$  for all  $t_0 > 0$  as claimed.  $\square$

## Step 7: Completing the proof.

**Lemma 7.3.28.** *Let  $F \in X^{\mathbf{p}}$  solve  $(\text{CR})_{DB}$  with  $\lim_{t \rightarrow \infty} F(t)_{\parallel} = 0$  in  $\mathcal{Z}'(\mathbb{R}^n : \mathbb{C}^{nm})$ . Then  $F = \tilde{F}$ .*

*Proof.* By Lemma 7.3.27 we have  $\partial_t F = \partial_t \tilde{F}$  in  $\mathcal{Z}'$ , so there exists  $G \in \mathcal{Z}'$  such that  $F(t_0) = G + \tilde{F}(t_0)$  for all  $t_0 \in \mathbb{R}_+$ . Since  $\lim_{t_0 \rightarrow \infty} \tilde{F}(t_0) = 0$  in  $\mathbf{X}_D^{\mathbf{p}}$  (Proposition 6.1.24, using the weak-star topology when  $\mathbf{p}$  is infinite) and hence also in  $\mathcal{Z}'$ , we find that  $G_{\parallel} = 0$ . Following the argument of [15, Step 5, page 50], we find that  $G = \beta a$  modulo polynomials, where  $a$  is invertible in  $L^{\infty}$  and  $\beta \in \mathbb{C}^m$ . To complete the proof it suffices to show that  $\beta = 0$ .

Note that the constant function  $[t \mapsto G = F(t) - \tilde{F}(t)]$  is in  $X^{\mathbf{p}}$ . If  $\mathbf{p}$  is finite, then  $G \in E^{\mathbf{p}}$  (since  $[t \mapsto G]$  solves  $(\text{CR})_{DB}$ ), and this forces  $\beta = 0$ . If  $\mathbf{p}$  is infinite, then the argument completing the proof of [15, Case  $q \leq 1$ , Theorem 1.3] shows that if  $\beta \neq 0$  then  $[t \mapsto G] \notin T_{-1; \tilde{\alpha}}^{\infty}$  for all  $\tilde{\alpha} \in [0, 1)$ . Since  $\theta(\mathbf{p}) > -1$  we have

$$G \in X^{\mathbf{p}} \hookrightarrow T_{-1; 1+\alpha(\mathbf{p})+\theta(\mathbf{p})}^{\infty},$$

and since  $\alpha(\mathbf{p}) + \theta(\mathbf{p}) \in [-1, 0)$  (this follows from  $\mathbf{p} \in I_{\max}$ ), we must have  $\beta = 0$ . This completes the proof.  $\square$

Therefore, by Corollary 7.3.26, under the assumptions of Theorem 7.3.2, we have that  $F = \tilde{F} = \mathbf{C}_{DB}^+(\tilde{F}(0))$ , with  $\tilde{F}(0) \in \mathbf{X}_D^{\mathbf{p},+}$  such that  $\|\tilde{F}(0)\|_{\mathbf{X}_D^{\mathbf{p},+}} \lesssim \|F\|_{X^{\mathbf{p}}}$  (by (7.38)). Furthermore, if  $f \in \mathbf{X}_{DB}^{\mathbf{p},+}$  and  $F = \mathbf{C}_{DB}^+ f$ , then by Proposition 6.1.24 we have

$$f = \lim_{t \rightarrow 0} \mathbf{C}_{DB}^+ F(t) = \tilde{F}(0)$$

with limit in  $\mathbf{X}_{DB}^{\mathbf{p}}$ . This completes the proof of Theorem 7.3.2.

## 7.4 Applications to boundary value problems

### 7.4.1 Characterisation of well-posedness and corollaries

First let us put the boundary value problems given in the introduction (Subsection 4.1.1) in a more convenient form.

Fix  $m \in \mathbb{N}$  and let  $\mathbf{p}$  be an exponent. Consider the spaces  $(\mathbf{X}^{\mathbf{p}} \cap D\mathcal{Z}')(\mathbb{R}^n : \mathbb{C}^{m(1+n)})$ , using the notation of Subsection 6.2.1. Making use of the natural splitting

$$\mathbf{X}^{\mathbf{p}}(\mathbb{R}^n : \mathbb{C}^{m(1+n)}) = \mathbf{X}^{\mathbf{p}}(\mathbb{R}^n : \mathbb{C}^m) \oplus \mathbf{X}^{\mathbf{p}}(\mathbb{R}^n : \mathbb{C}^{mn})$$

and the corresponding splitting for  $\mathcal{Z}'(\mathbb{R}^n : \mathbb{C}^{m(1+n)})$ , we can write

$$\begin{aligned} (\mathbf{X}^{\mathbf{p}} \cap D\mathcal{Z}')(\mathbb{R}^n : \mathbb{C}^{m(1+n)}) &= \mathbf{X}^{\mathbf{p}}(\mathbb{R}^n : \mathbb{C}^m) \oplus (\mathbf{X}^{\mathbf{p}}(\mathbb{R}^n : \mathbb{C}^{mn}) \cap \nabla_{\parallel} \mathcal{Z}'(\mathbb{R}^n : \mathbb{C}^{mn})) \\ &=: \mathbf{X}_{\perp}^{\mathbf{p}} \oplus \mathbf{X}_{\parallel}^{\mathbf{p}}. \end{aligned}$$

In particular, if  $\mathbf{p} \in I_{\max}$ , we can make the identification

$$\mathbf{X}_D^{\mathbf{p}}(\mathbb{R}^n : \mathbb{C}^{m(1+n)}) \simeq \mathbf{X}_{\perp}^{\mathbf{p}} \oplus \mathbf{X}_{\parallel}^{\mathbf{p}}$$

(see Theorem 6.2.1).

For  $\mathbf{p} \in I_{\max}$  with  $\theta(\mathbf{p}) < 0$ , define

$$\widetilde{X}^{\mathbf{p}} := \left\{ F \in X^{\mathbf{p}} : \lim_{t \rightarrow \infty} F(t)_{\parallel} = 0 \text{ in } \mathcal{Z}'(\mathbb{R}^n) \right\}$$

and when  $\mathbf{p} \in I_{\max}$  and  $\theta(\mathbf{p}) = 0$  (so  $\mathbf{p} = (p, 0)$  with  $p \in (n/(n+1), \infty)$ ) define

$$\widetilde{X}^{\mathbf{p}} := \left\{ F : N_*(F) \in L^{i(\mathbf{p})} \right\}$$

where  $N_*(F)$  is defined in (4.4).<sup>8</sup> We set  $\|F\|_{\widetilde{X}^{\mathbf{p}}}$  to be  $\|F\|_{X^{\mathbf{p}}}$  or  $\|N_*F\|_{L^{i(\mathbf{p})}}$  respectively.

**Definition 7.4.1.** For  $\mathbf{p} \in I_{\max}$  we define the *Regularity problem*

$$(R_{\mathbf{X}})_{\mathbf{A}}^{\mathbf{p}} : \begin{cases} L_A u = 0 & \text{in } \mathbb{R}_+^{1+n}, \\ \lim_{t \rightarrow 0} \nabla_{\parallel} u(t, \cdot) = f \in \mathbf{X}_{\parallel}^{\mathbf{p}}, \\ \|\nabla u\|_{\widetilde{X}^{\mathbf{p}}} \lesssim \|f\|_{\mathbf{X}^{\mathbf{p}}}, \end{cases}$$

and the *Neumann problem*

$$(N_{\mathbf{X}})_{\mathbf{A}}^{\mathbf{p}} : \begin{cases} L_A u = 0 & \text{in } \mathbb{R}_+^{1+n}, \\ \lim_{t \rightarrow 0} \partial_{\nu_A} u(t, \cdot) = f \in \mathbf{X}_{\perp}^{\mathbf{p}}, \\ \|\nabla u\|_{\widetilde{X}^{\mathbf{p}}} \lesssim \|f\|_{\mathbf{X}^{\mathbf{p}}}. \end{cases}$$

By  $\lim_{t \rightarrow 0} \nabla_{\parallel} u(t, \cdot) = f \in \mathbf{X}_{\parallel}^{\mathbf{p}}$  we mean that  $f \in \mathbf{X}_{\parallel}^{\mathbf{p}}$  and that the limit is in the  $\mathbf{X}_{\parallel}^{\mathbf{p}}$  topology, and likewise for the limit in the Neumann problem. We say that such a problem is *well-posed* if for all boundary data  $f$  there exists a unique  $u$  (up to additive constant) satisfying the conditions of the problem.

We will denote these problems simultaneously by  $(P_{\mathbf{X}})_{\mathbf{A}}^{\mathbf{p}}$ , with  $P$  standing for either  $R$  or  $N$ .

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<sup>8</sup>In the notation of Huang [51],  $\widetilde{X}^{(p,0)} = T_{\infty}^{p,2}$ .

*Remark 7.4.2.* The boundary condition in  $(R_{\mathbf{X}})_{\mathbf{A}}^{\mathbf{p}}$  is equivalent to the Dirichlet condition

$$\lim_{t \rightarrow 0} u(t, \cdot) = g \in \mathbf{X}_{\perp}^{\mathbf{p}+1}$$

where  $\nabla_{\parallel} g = f$ , since  $\nabla_{\parallel}$  is an isomorphism from  $\mathbf{X}_{\perp}^{\mathbf{p}+1}$  onto  $\mathbf{X}_{\parallel}^{\mathbf{p}}$ . Therefore  $(R_{\mathbf{X}})_{\mathbf{A}}^{\mathbf{p}}$  could be thought of as a Dirichlet problem  $(D_{\mathbf{X}})_{\mathbf{A}}^{\mathbf{p}+1}$ .

*Remark 7.4.3.* The problems  $(R_{\mathbf{X}})_{\mathbf{A}}^{\mathbf{p}}$  and  $(N_{\mathbf{X}})_{\mathbf{A}}^{\mathbf{p}}$  include all Regularity and Neumann problems introduced in Subsection 4.1.1. The definition above is much more concise (but, initially, much less clear).

Now we will use Theorems 7.3.1 and 7.3.2 to characterise the well-posedness of  $(R_{\mathbf{X}})_{\mathbf{A}}^{\mathbf{p}}$  and  $(N_{\mathbf{X}})_{\mathbf{A}}^{\mathbf{p}}$ . Let  $N_{\perp}$  and  $N_{\parallel}$  denote the projections from  $\mathbf{X}_D^{\mathbf{p}}(\mathbb{R}^n : \mathbb{C}^{m(1+n)})$  onto  $\mathbf{X}_{\perp}^{\mathbf{p}}$  and  $\mathbf{X}_{\parallel}^{\mathbf{p}}$  respectively. If  $\mathbf{p} \in I(\mathbf{X}, DB)$  or  $\mathbf{p}^{\heartsuit} \in I(\mathbf{X}, DB^*)$ , then we can realise  $\mathbf{X}_{DB}^{\mathbf{p},+}(\mathbb{R}^n : \mathbb{C}^{m(1+n)})$  as a subset of  $\mathbf{X}_D^{\mathbf{p}}(\mathbb{R}^n : \mathbb{C}^{m(1+n)})$ , and via this identification we define

$$N_{\mathbf{X},DB,\parallel}^{\mathbf{p}} : \mathbf{X}_{DB}^{\mathbf{p},+} \rightarrow \mathbf{X}_{\parallel}^{\mathbf{p}} \quad \text{and} \quad N_{\mathbf{X},DB,\perp}^{\mathbf{p}} : \mathbf{X}_{DB}^{\mathbf{p},+} \rightarrow \mathbf{X}_{\perp}^{\mathbf{p}}$$

Note again that the condition  $\mathbf{p}^{\heartsuit} \in I(\mathbf{X}, DB^*)$  is equivalent to  $\mathbf{p} \in I(\mathbf{X}, DB)$  when  $i(\mathbf{p}) \in (1, \infty)$ .

**Theorem 7.4.4** (Characterisation of well-posedness). *Let  $B = \hat{A}$ . Suppose  $\mathbf{p}$  satisfies*

$$\begin{cases} \mathbf{p} \in I(\mathbf{X}, DB) & \text{if } i(\mathbf{p}) \leq 2, \\ \mathbf{p}^{\heartsuit} \in I(\mathbf{X}, DB^*) & \text{if } i(\mathbf{p}) > 2. \end{cases}$$

*Then  $(R_{\mathbf{X}})_{\mathbf{A}}^{\mathbf{p}}$  (resp.  $(N_{\mathbf{X}})_{\mathbf{A}}^{\mathbf{p}}$ ) is well-posed if and only if  $N_{\mathbf{X},DB,\parallel}^{\mathbf{p}}$  (resp.  $N_{\mathbf{X},DB,\perp}^{\mathbf{p}}$ ) is an isomorphism.*

*Proof.* The results for  $\theta(\mathbf{p}) = 0$  and  $\theta(\mathbf{p}) = -1$  correspond to [15, Theorems 1.5 and 1.6], so we need only consider  $\theta(\mathbf{p}) \in (-1, 0)$ . Since  $\nabla_A = [\partial_{\nu_A}, \nabla_{\parallel}]$ , The boundary conditions for  $(R_{\mathbf{X}})_{\mathbf{A}}^{\mathbf{p}}$  and  $(N_{\mathbf{X}})_{\mathbf{A}}^{\mathbf{p}}$  can be rewritten as

$$\begin{aligned} N_{\parallel} \left( \lim_{t \rightarrow 0} \nabla_A u(t, \cdot) \right) &= f \in \mathbf{X}_{\parallel}^{\mathbf{p}} \quad \text{and} \\ N_{\perp} \left( \lim_{t \rightarrow 0} \nabla_A u(t, \cdot) \right) &= f \in \mathbf{X}_{\perp}^{\mathbf{p}} \end{aligned}$$

respectively. By Theorem 4.1.3, solutions  $u$  to  $L_A u = 0$  are in bijective correspondence (modulo additive constant) to solutions  $F$  to  $(\text{CR})_{DB}$ , with  $F = \nabla_A u$ . Furthermore, by Theorems 7.3.1 and 7.3.2 and by the assumptions of this theorem, every such  $F \in \widetilde{X}^{\mathbf{p}}$  is given by  $F = \mathbf{C}_{DB}^+ F_0$  for a unique  $F_0 \in \mathbf{X}_{DB}^{\mathbf{p},+}$  (and so

$F(t) \in \mathbf{X}_{DB}^{\mathbf{p},+}$  for all  $t$ ), every such  $F_0$  determines a solution  $F$ , and by continuity of the semigroup on  $\mathbf{X}_{DB}^{\mathbf{p},+}$  (Proposition 6.1.24)

$$F_0 = \lim_{t \rightarrow 0} \nabla_A u(t, \cdot).$$

The result follows.  $\square$

Define the *energy exponent*  $\mathbf{e} = (2, -1/2)$ . For all  $A$ , the Lax–Milgram theorem guarantees well-posedness of the problems  $(R_{\mathbf{X}})_{\mathbf{A}}^{\mathbf{e}}$  and  $(N_{\mathbf{X}})_{\mathbf{A}}^{\mathbf{e}}$  (see [12, Theorems 3.2 and 3.3]). We say that a problem  $(P_{\mathbf{X}})_{\mathbf{A}}^{\mathbf{p}}$  is *compatibly well-posed* if it is well-posed and if for all boundary data  $f \in \mathbf{X}_{\bullet}^{\mathbf{p}} \cap \mathbf{X}_{\bullet}^{\mathbf{e}}$  (where  $\bullet$  is either  $\parallel$  or  $\perp$  depending on the choice of boundary condition), the solution to  $(P_{\mathbf{X}})_{\mathbf{A}}^{\mathbf{e}}$  with boundary data  $f$  (the *energy solution*) coincides with the solution to  $(P_{\mathbf{X}})_{\mathbf{A}}^{\mathbf{p}}$  with boundary data  $f$ . If  $\mathbf{p}$  satisfies the assumptions of Theorem 7.4.4, then this theorem says that  $(P_{\mathbf{X}})_{\mathbf{A}}^{\mathbf{p}}$  is compatibly well-posed if and only if  $N_{\mathbf{X},DB,\bullet}^{\mathbf{p}}$  is an isomorphism and  $(N_{\mathbf{X},DB,\bullet}^{\mathbf{p}})^{-1} = (N_{\mathbf{X},DB,\bullet}^{\mathbf{e}})^{-1}$  on  $\mathbf{X}_{\bullet}^{\mathbf{p}} \cap \mathbf{X}_{\bullet}^{\mathbf{e}}$ .

For finite exponents we can interpolate compatible well-posedness; compatibility is required in order to interpolate invertibility.

**Theorem 7.4.5** (Interpolation of compatible well-posedness). *Fix  $\theta \in (0, 1)$ , and suppose  $\mathbf{p}$  and  $\mathbf{q}$  are finite exponents satisfying the assumptions of Theorem 7.4.4. If  $(P_{\mathbf{X}})_{\mathbf{A}}^{\mathbf{p}}$  and  $(P_{\mathbf{X}})_{\mathbf{A}}^{\mathbf{q}}$  are compatibly well-posed, then  $(P_{\mathbf{X}})_{\mathbf{A}}^{[\mathbf{p},\mathbf{q}]_{\theta}}$  is compatibly well-posed. Furthermore, if  $\theta(\mathbf{p}) \neq \theta(\mathbf{q})$  and  $\mathbf{X} = \mathbf{H}$ , then  $(P_{\mathbf{B}})_{\mathbf{A}}^{[\mathbf{p},\mathbf{q}]_{\theta}}$  is also compatibly well-posed.*

*Proof.* We use interpolation result for smoothness spaces, Theorem 5.1.52. By the previous discussion, we have

$$(N_{\mathbf{X},DB,\bullet}^{\mathbf{p}})^{-1} = (N_{\mathbf{X},DB,\bullet}^{\mathbf{q}})^{-1} = (N_{\mathbf{X},DB,\bullet}^{\mathbf{e}})^{-1}$$

on the intersection  $\mathbf{X}_{\bullet}^{\mathbf{p}} \cap \mathbf{X}_{\bullet}^{\mathbf{q}} \cap \mathbf{X}_{\bullet}^{\mathbf{e}}$ . Since this intersection is dense in  $\mathbf{X}_{\bullet}^{\mathbf{p}}$  and  $\mathbf{X}_{\bullet}^{\mathbf{q}}$  (here is where we use finiteness of  $\mathbf{p}$  and  $\mathbf{q}$ ), we have a well-defined operator

$$\mathbf{N}: \mathbf{X}_{\bullet}^{\mathbf{p}} + \mathbf{X}_{\bullet}^{\mathbf{q}} \rightarrow \mathbf{X}_{DB}^{\mathbf{p},+} + \mathbf{X}_{DB}^{\mathbf{q},+}$$

which restricts to  $(N_{\mathbf{X},DB,\bullet}^{\mathbf{p}})^{-1}$  and  $(N_{\mathbf{X},DB,\bullet}^{\mathbf{q}})^{-1}$  on  $\mathbf{X}_{\bullet}^{\mathbf{p}}$  and  $\mathbf{X}_{\bullet}^{\mathbf{q}}$  respectively. By complex interpolation,  $\mathbf{N}$  restricts to a bounded operator  $\mathbf{N}_{\theta}: \mathbf{X}_{\bullet}^{[\mathbf{p},\mathbf{q}]_{\theta}} \rightarrow \mathbf{X}_{DB}^{[\mathbf{p},\mathbf{q}]_{\theta},+}$ . Since  $\mathbf{N}_{\theta}$  is equal to  $(N_{\mathbf{X},DB,\bullet}^{\mathbf{e}})^{-1}$  on  $\mathbf{X}_{\bullet}^{[\mathbf{p},\mathbf{q}]_{\theta}} \cap \mathbf{X}_{\bullet}^{\mathbf{e}}$ , and since  $N_{\mathbf{X},DB,\bullet}^{[\mathbf{p},\mathbf{q}]_{\theta}}$  is equal to  $N_{\mathbf{X},DB,\bullet}^{\mathbf{e}}$  on  $\mathbf{X}_{DB}^{\mathbf{p},+} \cap \mathbf{X}_{DB}^{\mathbf{q},+}$ , we find that  $\mathbf{N}_{\theta}$  is the inverse of  $N_{\mathbf{X},DB,\bullet}^{[\mathbf{p},\mathbf{q}]_{\theta}}$ . Therefore  $N_{\mathbf{X},DB,\bullet}^{[\mathbf{p},\mathbf{q}]_{\theta}}$  is an isomorphism, and since  $[\mathbf{p}, \mathbf{q}]_{\theta}$  satisfies the assumptions of Theorem

**7.4.4** (by Proposition 6.2.9),  $(P_{\mathbf{X}})_A^{[\mathbf{p}, \mathbf{q}]_\theta}$  is well-posed. Furthermore, since  $\mathbf{N}_\theta = (N_{\mathbf{X}, DB, \bullet}^e)^{-1}$  on  $\mathbf{X}_\bullet^{[\mathbf{p}, \mathbf{q}]_\theta} \cap \mathbf{X}_\bullet^e$ ,  $(P_{\mathbf{X}})_A^{[\mathbf{p}, \mathbf{q}]_\theta}$  is compatibly well-posed. When  $\mathbf{X} = \mathbf{H}$  and  $\theta(\mathbf{p}) \neq \theta(\mathbf{q})$ , applying real interpolation with the same argument yields compatible well-posedness of  $(P_{\mathbf{B}})_A^{[\mathbf{p}, \mathbf{q}]_\theta}$ .  $\square$

Although well-posedness without compatibility cannot be interpolated, it can be extrapolated by making use of a theorem of Šneřberg [83].<sup>9</sup> This extrapolation procedure also extrapolates compatible well-posedness, and works for infinite exponents (excluding the BMO-Sobolev range of spaces).

**Theorem 7.4.6** (Extrapolation of well-posedness). *Let  $B = \hat{A}$ , and let  $\mathbf{p}$  satisfy*

$$\begin{cases} \mathbf{p} \in I(\mathbf{X}, DB)^\circ & i(\mathbf{p}) \leq 2 \\ \mathbf{p}^\heartsuit \in I(\mathbf{X}, DB^*)^\circ & i(\mathbf{p}) > 2 \end{cases}$$

(note the appearance of the interior of the identification regions), and if  $\mathbf{X} = \mathbf{H}$  then further assume that  $j(\mathbf{p}) \neq 0$ . Suppose also that  $(P_{\mathbf{X}})_A^{\mathbf{p}}$  is (compatibly) well-posed. Then there exists a  $(j, \theta)$ -neighbourhood  $O_{\mathbf{p}}$  of  $\mathbf{p}$  such that for all  $\mathbf{q} \in O_{\mathbf{p}}$ ,  $(P_{\mathbf{X}})_A^{\mathbf{q}}$  is (compatibly) well-posed.

The restriction  $j(\mathbf{p}) \neq 0$  for  $\mathbf{X} = \mathbf{H}$  rules out BMO-Sobolev spaces, which are not in the interior of any of our complex interpolation scales. Note that when  $\mathbf{p} \in (1, \infty)$ ,  $\mathbf{p}^\heartsuit \in I(\mathbf{X}, DB^*)^\circ$  is equivalent to  $\mathbf{p} \in I(\mathbf{X}, DB)^\circ$ .

*Proof.* We will prove the result for  $i(\mathbf{p}) \in (1, \infty)$  as the proof for general exponents follows the same argument.

Let  $\bullet$  denote either  $\perp$  or  $\parallel$  as before. By Theorem 7.4.4,  $N_{\mathbf{X}, DB, \bullet}^{\mathbf{p}}: \mathbf{X}_{DB}^{\mathbf{p}, +} \rightarrow \mathbf{X}_\bullet^{\mathbf{p}}$  is an isomorphism. Let  $B_{\mathbf{p}}$  be a ball in the  $(j, \theta)$ -plane centred at  $\mathbf{p}$  such that  $B_{\mathbf{p}} \subset I(\mathbf{X}, DB)$ . Fix  $\mathbf{r} \in B_{\mathbf{p}}$ . Then we have

$$\mathbf{X}_{DB}^{\mathbf{p}, +} = [\mathbf{X}_{DB}^{[\mathbf{p}, \mathbf{r}]^{-1}, +}, \mathbf{X}_{DB}^{\mathbf{r}, +}]_{1/2}$$

since  $\mathbf{p} = [[\mathbf{p}, \mathbf{r}]_{-1}, \mathbf{r}]_{1/2}$ . Since the spaces  $\mathbf{X}_\bullet^{\mathbf{p}}$  form a complex interpolation scale,<sup>10</sup> by the extrapolation theorem of Šneřberg,<sup>11</sup> there exists  $\varepsilon > 0$  such that

$$N_{\mathbf{X}, DB, \bullet}^{[\mathbf{p}, \mathbf{r}]_\nu}: \mathbf{X}_{DB}^{[\mathbf{p}, \mathbf{r}]_\nu, +} \rightarrow \mathbf{X}_\bullet^{[\mathbf{p}, \mathbf{r}]_\nu} \quad \text{is an isomorphism for all } \nu \in (-\varepsilon, \varepsilon).$$

<sup>9</sup>This was extended to quasi-Banach spaces by Kalton and Mitrea [58, Theorem 2.7], and elaborated upon by Kalton, Mayboroda, and Mitrea [57, Theorem 8.1].

<sup>10</sup>This is immediate for  $\mathbf{X}_\perp^{\mathbf{p}}$ , and for  $\mathbf{X}_\parallel^{\mathbf{p}}$  this is because  $\mathbf{X}_\parallel^{\mathbf{p}}$  is the image of  $\mathbf{X}^{\mathbf{p}}(\mathbb{R}^n: \mathbb{C}^{m(1+n)})$  under the retraction  $N_{\parallel} \mathbb{P}_D$ .

<sup>11</sup>See [58, Theorem 2.7] for a reference incorporating both quasi-Banach spaces and the English language.



Furthermore, inspection of the Kalton–Mitrea proof of this result shows that  $\varepsilon$  is independent of  $\mathbf{r}$ . Therefore there exists a ball  $O_{\mathbf{p}} \subset B_{\mathbf{p}}$  centred at  $\mathbf{p}$  such that

$$N_{\mathbf{X},DB,\bullet}^{\mathbf{q}}: \mathbf{X}_{DB}^{\mathbf{q},+} \rightarrow \mathbf{X}_{\bullet}^{\mathbf{q}} \quad \text{is an isomorphism for all } \mathbf{q} \in O_{\mathbf{p}}.$$

In addition, the inverses of these maps are consistent ([57, Theorem 8.1]), and so if the inverse of  $N_{\mathbf{X},DB,\bullet}^{\mathbf{p}}$  is consistent with that of  $N_{\mathbf{X},DB,\bullet}^{\mathbf{e}}$  (i.e. when  $(P_{\mathbf{X}})_{\mathbf{A}}^{\mathbf{p}}$  is compatibly well-posed) then this also holds for all  $\mathbf{q} \in O_{\mathbf{p}}$ . By Theorem 7.4.4, this completes the proof.  $\square$

*Remark 7.4.7.* Note that this proof also shows that if  $(P_{\mathbf{X}})_{\mathbf{A}}^{\mathbf{p}}$  is well-posed but not compatibly well-posed, then the same is true for  $(P_{\mathbf{X}})_{\mathbf{A}}^{\mathbf{q}}$  for nearby  $\mathbf{q}$  (the inverses of  $N_{\mathbf{X},DB,\bullet}^{\mathbf{q}}$  are consistent, so they are either all consistent with  $N_{\mathbf{X},DB,\bullet}^{\mathbf{e}}$  or all not consistent with  $N_{\mathbf{X},DB,\bullet}^{\mathbf{e}}$ ). Therefore, staying within the range of exponents for which Theorem 7.4.4 holds, the set of  $\mathbf{p}$  such that  $(P_{\mathbf{X}})_{\mathbf{A}}^{\mathbf{p}}$  is compatibly well-posed is a connected component of the set of  $\mathbf{p}$  such that  $(P_{\mathbf{X}})_{\mathbf{A}}^{\mathbf{p}}$  is well-posed.

Now we present a  $\heartsuit$ -duality principle for well-posedness.

**Theorem 7.4.8** ( $\heartsuit$ -duality of well-posedness). *Let  $B = \hat{A}$ , and suppose that  $\mathbf{p} \in I(\mathbf{X}, DB)$ . If  $(P_{\mathbf{X}})_{\mathbf{A}}^{\mathbf{p}}$  is (compatibly) well-posed, then  $(P_{\mathbf{X}})_{\mathbf{A}^*}^{\heartsuit}$  is also (compatibly) well-posed.*

Of course, if  $i(\mathbf{p}) \in (1, \infty)$ , then this statement is an equivalence. We point out the case where  $\mathbf{p} = (1, s)$  with  $s \in (-1, 0]$ : in this case the result says that well-posedness of a problem with coefficients  $A$  and boundary data in the Hardy–Sobolev space  $\dot{H}_s^1$  (resp. the Besov space  $\dot{B}_s^{1,1}$ ) implies well-posedness of the corresponding problem for  $A^*$  with boundary data in the image of BMO–Sobolev space  $B\dot{M}O_{-s}$  (resp. the Hölder space  $\dot{\Lambda}_{-s}$ ) under  $D$ .

*Proof.* We will be sketchy because all the important details of this argument are already done by Auscher, Mourgoglou, and Stahlhut (see [16, §12.2] and [15, §13]). Recall from Remark 6.2.11 that  $\widehat{A}^* = NB^*N =: \tilde{B}$ . When  $\mathbf{p}$  is finite, the pairing

$$\langle f, g \rangle_{\mathbb{X}_{DB}^{\mathbf{p}}}^N := \langle f, Ng \rangle_{\mathbb{X}_{DB}^{\mathbf{p}}}$$

a duality pairing between  $\mathbb{X}_{DB}^{\mathbf{p}}$  and  $\mathbb{X}_{\tilde{B}D}^{\mathbf{p}'}$ . We have that  $\|Dg\|_{\mathbb{X}_{D\tilde{B}}^{\mathbf{p}}}^{\heartsuit} \simeq \|g\|_{\mathbb{X}_{\tilde{B}D}^{\mathbf{p}'}}$  whenever  $g \in \mathcal{D}(D) \cap \mathbb{X}_{\tilde{B}D}^{\mathbf{p}'}$  (Proposition 6.2.6), and so the pairing

$$\langle f, g \rangle_{\mathbb{X}_{DB}^{\mathbf{p}}}^{\heartsuit} := \langle f, ND^{-1}g \rangle_{\mathbb{X}_{DB}^{\mathbf{p}}} \quad (7.50)$$

is a duality pairing between  $\mathbb{X}_{DB}^{\mathbf{p}}$  and  $\mathcal{R}(D) \cap \mathbb{X}_{D\tilde{B}}^{\mathbf{p}\heartsuit}$ . Since  $\mathbf{p} \in I(\mathbf{X}, DB)$ , we can identify  $\mathbf{X}_D^{\mathbf{p}} = \mathbf{X}_{DB}^{\mathbf{p}}$  and  $\mathbf{X}_D^{\mathbf{p}\heartsuit} = \mathbf{X}_{D\tilde{B}}^{\mathbf{p}\heartsuit}$  as completions of  $\mathbb{X}_{DB}^{\mathbf{p}}$  and  $\mathbb{X}_{D\tilde{B}}^{\mathbf{p}\heartsuit}$  respectively (using a simple modification of Proposition 6.2.7 to make the second identification), and by density the pairing (7.50) extends to a duality pairing between  $\mathbf{X}_{DB}^{\mathbf{p}}$  and  $\mathbf{X}_{D\tilde{B}}^{\mathbf{p}\heartsuit}$ . As in the proof of [15, Lemma 13.3], this pairing realises  $\mathbf{X}_{D\tilde{B}}^{\mathbf{p}\heartsuit, \mp}$  as the dual of  $\mathbf{X}_{DB}^{\mathbf{p}, \pm}$ ,  $\mathbf{X}_{\perp}^{\mathbf{p}\heartsuit}$  as the dual of  $\mathbf{X}_{\parallel}^{\mathbf{p}}$ , and  $\mathbf{X}_{\parallel}^{\mathbf{p}\heartsuit}$  as the dual of  $\mathbf{X}_{\perp}^{\mathbf{p}}$ . The remainder of the argument precisely follows the proof of [15, Theorem 1.6].  $\square$

### 7.4.2 The regularity problem for real coefficient scalar equations

The results above show that from compatible well-posedness of a boundary value problem for an exponent  $\mathbf{p} \in I(\mathbf{X}, DB)$  with  $B = \hat{A}$ , we may deduce compatible well-posedness for a larger range of exponents by  $\heartsuit$ -duality and interpolation. As an application of this principle we consider the regularity problems  $(R_{\mathbf{X}})_{\mathbf{A}}^{\mathbf{p}}$  in the real scalar case.

Suppose that  $m = 1$  (so that  $L_A u = 0$  is a single equation rather than a system) and that the entries of  $A$  are real. In this setting, there exists a number  $\alpha \in (0, 1]$  such that for every Euclidean ball  $B = B(X_0, 2r)$  in  $\mathbb{R}_+^{1+n}$  and every solution  $u$  to  $L_A u = 0$  in  $B$ , we have

$$|u(X) - u(X')| \lesssim \left( \frac{|X - X'|}{r} \right)^{\alpha} \left( \iint_B |u|^2 \right)^{1/2} \quad (7.51)$$

for all  $X, X'$  in the smaller ball  $B(X_0, r)$ . In this case say that the coefficients  $A$  satisfy the *De Giorgi–Nash–Moser condition of exponent  $\alpha$* . The adjoint matrix  $A^*$  will also satisfy a De Giorgi–Nash–Moser condition of (possibly different) exponent  $\alpha^*$ .

Auscher and Stahlhut [16, Corollary 13.3] show that in this case<sup>12</sup> we have

$$\begin{aligned} \left( \frac{n}{n + \alpha}, p_+(DB) \right) &\subset I_0(\mathbf{H}, DB), \\ \left( \frac{n}{n + \alpha^*}, p_+(D\tilde{B}) \right) &\subset I_0(\mathbf{H}, D\tilde{B}), \end{aligned}$$

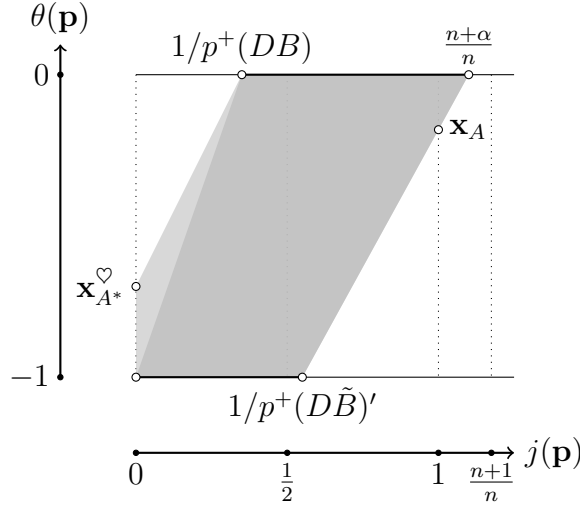
where  $\tilde{B} = \hat{A}^*$  (note that  $\tilde{B} \neq B^*$ ) and where  $p_+(DB), p_+(D\tilde{B}) > 2$ . Therefore

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<sup>12</sup>In fact, a somewhat weaker assumption is needed there.



Figure 7.6: More exponents  $\mathbf{p} \in I(\mathbf{H}, DB)$ , when  $m = 1$  and  $A$  is real, with  $B = \hat{A}$ . The dark shaded region corresponds to Figure 7.5.



previous argument to deduce compatible well-posedness of  $(R_{\mathbf{H}})_{\mathbf{A}}^{\mathbf{p}}$  for  $\mathbf{p}$  in the region pictured in Figure 7.7, and of  $(R_{\mathbf{B}})_{\mathbf{A}}^{\mathbf{p}}$  in the interior of this region.<sup>15</sup>

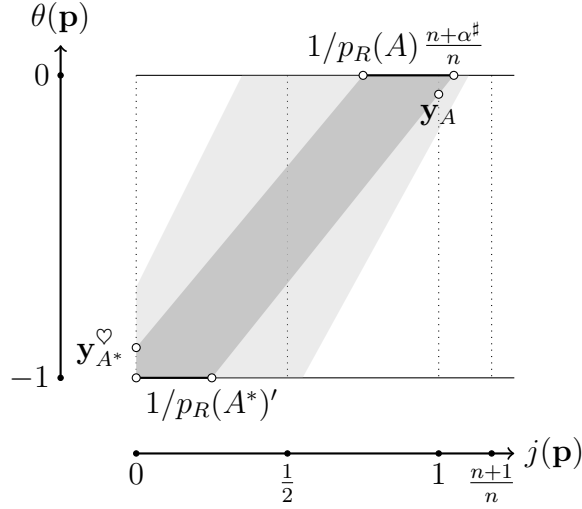
We can expand this region slightly for Besov spaces: applying  $\heartsuit$ -duality to compatible well-posedness of  $(R_{\mathbf{B}})_{\mathbf{A}^*}^{\mathbf{p}}$  for  $\mathbf{p}$  in the open triangle with vertices  $\mathbf{y}_{A^*}$ ,  $(n + \alpha^\sharp/n, 0)$ , and  $(1, 0)$ , we find that  $(R_{\mathbf{B}})_{\mathbf{A}}^{(\infty, \alpha; 0)}$  is compatibly well-posed for all  $\alpha \in (-1, -1 - \alpha^\sharp)$ . Therefore (after another iteration of interpolation) we have well-posedness of  $(R_{\mathbf{B}})_{\mathbf{A}}^{\mathbf{p}}$  for all  $\mathbf{p}$  in the shaded region of Figure 7.8. This is the same region obtained by Barton and Mayboroda for compatible well-posedness of  $(R_{\mathbf{B}})_{\mathbf{A}}^{\mathbf{p}}$  in this setting [21, Figure 3.5].<sup>16</sup> To recover the result of [21, Corollary 3.24], one need only apply Lemma 7.2.1 (which is valid for this region of  $\mathbf{p}$ , see Figure 7.2) to remove the decay assumption at infinity from  $(R_{\mathbf{B}})_{\mathbf{A}}^{\mathbf{p}}$ , and the trace theorem [21, Theorem 6.3] to replace our boundary condition with a trace condition.

In the case that  $A$  is symmetric in addition to the above assumptions, then results of Kenig and Pipher [61] imply that we have the additional information

<sup>15</sup>We can also deduce results for  $BMO$ -Sobolev spaces, which correspond to the unpictured  $j(\mathbf{p}) = 0$  range.

<sup>16</sup>The only difference is in the light shaded ‘region of applicability’: ours depends on the Auscher–Stahlhut exponent  $p^+(DB)$ , while that in Barton–Mayboroda is in terms of the exponent appearing in Meyers’ theorem [75, Theorem 2] (see also [21, Lemma 2.12]). It is not clear whether there is any relationship between these exponents.

Figure 7.7: Exponents  $\mathbf{p}$  for which  $(R_{\mathbf{H}})_{\mathbf{A}}^{\mathbf{p}}$  is compatibly well-posed (the dark shaded region). The light shaded region is from Figure 7.6.



$p_R(A) = p_R(A^*) > 2$ , and regions of compatible well-posedness of  $(R_{\mathbf{X}})_{\mathbf{A}}^{\mathbf{p}}$  can be expanded accordingly. Furthermore, in this case the corresponding Neumann problems  $(N_{\mathbf{X}})_{\mathbf{A}}^{\mathbf{p}}$  are well-posed for the same range of  $\mathbf{p}$  by repeating the arguments above (starting from the information given by [61]).

### 7.4.3 Additional boundary behaviour of solutions

It is possible to establish the following boundary behaviour of solutions to  $L_A u = 0$ .

**Theorem 7.4.9.** *Let  $B = \hat{A}$  and let  $\mathbf{p}$  be an exponent with  $\theta(\mathbf{p}) \in (-1, 0)$ . Let  $u$  solve  $L_A u = 0$ , with  $\nabla_A u \in \widetilde{X^{\mathbf{p}}}$ .*

(i) *Suppose  $\mathbf{p}$  is finite and  $\mathbf{p} \in I(\mathbf{X}, DB)$ . Then there exists  $v \in \mathbf{X}^{\mathbf{p}+1}$  such that*

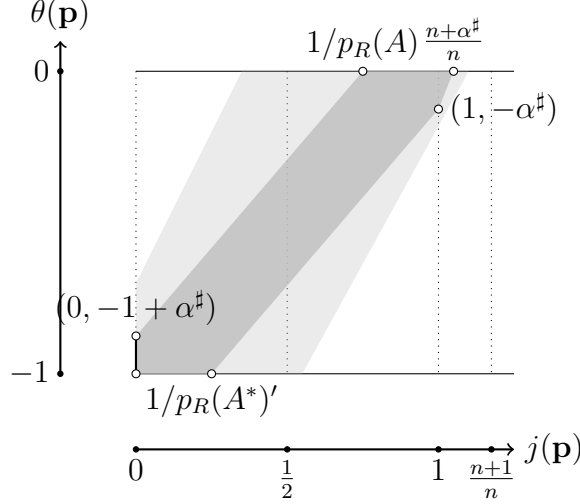
$$\lim_{R \rightarrow 0} \iint_{\Omega(R, x)} u(\tau, \xi) d\xi d\tau = v(x) \quad \text{a.e. } x \in \mathbb{R}^n,$$

*with  $\nabla_{\parallel} u = \nabla_{\parallel} v$  in  $\mathcal{Z}'$ .*

(ii) *Suppose  $\mathbf{p}$  is infinite and  $\mathbf{p}^{\heartsuit} \in I(\mathbf{X}, DB^*)$ . Then  $u \in \mathbf{X}^{\mathbf{p}+1}(\overline{\mathbb{R}_+^{1+n}})$ .*

The proof, which we do not provide here, requires a series of *ad hoc* arguments (much like the proof of Theorem 6.2.12) that exploit the semigroup representation

Figure 7.8: Exponents  $\mathbf{p}$  for which  $(R_{\mathbf{B}})_{\mathbf{A}}^{\mathbf{p}}$  is compatibly well-posed (the dark shaded region); this includes no exponents with  $\theta(\mathbf{p}) = 0$  or  $\theta(\mathbf{p}) = -1$ . The light shaded region is from Figure 7.6.



of the conormal gradient  $\nabla_A u$  provided by Theorems 7.3.1 and 7.3.2. Full details, which have been communicated to us by Pascal Auscher, will be provided in a future version of this article.

#### 7.4.4 Layer potentials

We conclude the article by briefly indicating the relation between the first-order approach and the method of layer potentials. Further information on this link is available in [79] and [16, §12.3].

Suppose, for the moment, that  $A$  and  $A^*$  both satisfy the De Giorgi–Nash–Moser condition (7.51) of some exponent. Then for all  $(t, x) \in \mathbb{R}^{1+n}$  there exists a fundamental solution  $\Gamma_{(t,x)}$  for  $L_{A^*}$  in  $\mathbb{R}^{1+n}$  with pole at  $(t, x)$ .<sup>17</sup> The fundamental solution  $\Gamma_{(t,x)}$  is a  $\mathbb{C}^m$ -valued function on  $\mathbb{R}_+^{1+n}$  satisfying

$$\operatorname{div} A^* \nabla \Gamma_{(t,x)} = \delta_{(t,x)} \mathbf{1} \quad \text{in } \mathbb{R}^{1+n}$$

in the usual weak sense, where  $\delta_{(t,x)}$  is the Dirac mass at  $(t, x)$  and  $\mathbf{1} = (1, \dots, 1) \in \mathbb{C}^m$ .

<sup>17</sup>Fundamental solutions were constructed in dimension  $n+1 \geq 3$  by Hofmann and Kim [47], and in dimension  $n+1 = 2$  by Rosén [79].

For a (reasonable) function  $h: \mathbb{R}^n \rightarrow \mathbb{C}^m$  and for  $(t, x) \in \mathbb{R}_+^{1+n}$ , define the *double layer potential*

$$\mathcal{D}_t h(x)^i := \int_{\mathbb{R}^n} \left( \partial_{\nu_A^*} \Gamma_{(t,x)}(0, y)^i, h(y) \right) dy \quad (i = 1, \dots, m).$$

and the *single layer potential*

$$\mathcal{S}_t h(x)^i := \int_{\mathbb{R}^n} (\Gamma_{(t,x)}(0, y)^i, h(y)) dy. \quad (i = 1, \dots, m).$$

One can solve Dirichlet problems for  $L_A$  in  $\mathbb{R}_+^{1+n}$  with boundary data  $\varphi$  by solving the double layer equation

$$\lim_{t \searrow 0} \mathcal{D}_t h = \varphi,$$

and likewise one can solve Neumann problems for  $L_A$  in  $\mathbb{R}_+^{1+n}$  with boundary data  $\varphi$  by solving the single layer equation

$$\lim_{t \searrow 0} \partial_{\nu_A} \mathcal{S}_t h = \varphi.$$

The corresponding solutions  $u$  are then given by  $u(t, x) = \mathcal{D}_t h(x)$  and  $u(t, x) = \mathcal{S}_t h(x)$  respectively.

It was shown by Rosén [79] that these layer potential operators fall within the scope of the first-order framework. Keeping the De Giorgi–Nash–Moser assumption on  $A$  and  $A^*$ , and writing  $B = \hat{A}$  as usual, for all  $f \in L^2(\mathbb{R}^n : \mathbb{C}^m)$  and  $t \in \mathbb{R}$  we have

$$\mathcal{D}_t f = \operatorname{sgn}(t) \left( e^{-|t|BD} \chi^{\operatorname{sgn}(t)}(BD) \begin{bmatrix} f \\ 0 \end{bmatrix} \right)_{\perp}$$

and

$$\nabla_A \mathcal{S}_t f = -\operatorname{sgn}(t) \left( e^{-|t|DB} \chi^{\operatorname{sgn}(t)}(DB) \begin{bmatrix} f \\ 0 \end{bmatrix} \right)_{\perp},$$

where the vectors  $\begin{bmatrix} f \\ 0 \end{bmatrix}$  are in  $L^2(\mathbb{R}^n : \mathbb{C}^{m(1+n)})$ , written with respect to the transversal/tangential splitting. In terms of Cauchy operators, on  $\mathbb{R}_{\pm}^{1+n}$  we can write

$$\mathcal{D}f = \pm \left( C_{BD}^{\pm} \begin{bmatrix} f \\ 0 \end{bmatrix} \right)_{\perp}, \quad \nabla_A \mathcal{S}f = \mp \left( C_{DB}^{\pm} \begin{bmatrix} f \\ 0 \end{bmatrix} \right)_{\perp}. \quad (7.52)$$

The right hand sides of these expressions are defined for all coefficients  $A$ , whether or not the De Giorgi–Nash–Moser assumptions are satisfied.

For all exponents  $\mathbf{p} \in I(\mathbf{X}, DB)$  (and for all infinite exponents  $\mathbf{p}$  with  $\mathbf{p}^\heartsuit \in I(\mathbf{X}, DB^*)$ ) we have the estimate

$$\|C_{DB}^+ f\|_{X^{\mathbf{p}}} \lesssim \|f\|_{\mathbb{X}_D^{\mathbf{p}}} \quad (f \in \overline{\mathcal{R}(DB)}^+)$$

(see Theorems 6.1.25 and 6.2.12), which immediately yields

$$\|\nabla_A \mathcal{S}f\|_{X^{\mathbf{p}}} \lesssim \|f\|_{X^{\mathbf{p}}}. \quad (7.53)$$

This can be seen as boundedness of  $\mathcal{S}$  from the classical smoothness space  $\mathbf{X}^{\mathbf{p}}$  into a Sobolev-type space built on  $X^{\mathbf{p}}$ . On the other hand, because of the equality  $\nabla_{\parallel} g_{\perp} = -(Dg)_{\parallel}$  (for any  $g: \mathbb{R}^n \rightarrow \mathbb{C}^{m(1+n)}$ ) and similarity of functional calculus, we can write

$$\nabla_{\parallel} \mathcal{D}f = - \left( C_{DB}^+ D \begin{bmatrix} f \\ 0 \end{bmatrix} \right)_{\parallel} = \left( C_{DB}^+ \begin{bmatrix} 0 \\ \nabla_{\parallel} f \end{bmatrix} \right)_{\parallel},$$

which implies (for all  $f$  with  $\nabla_{\parallel} f \in L^2$ )

$$\begin{aligned} \|\nabla_{\parallel} \mathcal{D}f\|_{X^{\mathbf{p}}} &\lesssim \left\| C_{DB}^+ \begin{bmatrix} 0 \\ \nabla_{\parallel} f \end{bmatrix} \right\|_{X^{\mathbf{p}}} \\ &\lesssim \|\nabla_{\parallel} f\|_{X^{\mathbf{p}}} \\ &\simeq \|f\|_{X^{\mathbf{p}+1}}, \end{aligned}$$

and similarly

$$\begin{aligned} \|\partial_t \mathcal{D}f\|_{X^{\mathbf{p}}} &\lesssim \left\| BDC_{BD}^+ \begin{bmatrix} f \\ 0 \end{bmatrix} \right\|_{X^{\mathbf{p}}} \\ &\lesssim \left\| C_{DB}^+ D \begin{bmatrix} f \\ 0 \end{bmatrix} \right\|_{X^{\mathbf{p}}} \\ &\lesssim \|f\|_{X^{\mathbf{p}+1}}, \end{aligned}$$

so that

$$\|\nabla \mathcal{D}f\|_{X^{\mathbf{p}}} \lesssim \|f\|_{X^{\mathbf{p}+1}}. \quad (7.54)$$

Bounds for layer potentials on the lower half-space corresponding to (7.53) and (7.54) can also be derived. Compare these results with those of Barton and Mayboroda [21, Theorem 3.1]. Various other mapping properties of layer potentials follow from the identifications (7.52) and the mapping properties of functional calculus on the spaces  $\mathbf{X}_{DB}^{\mathbf{p}}$ , for example the uniform bounds

$$\sup_{t \neq 0} \|\nabla_A \mathcal{S}_t f\|_{X^{\mathbf{p}}} + \|\mathcal{S}_t f\|_{X^{\mathbf{p}+1}} \lesssim \|f\|_{X^{\mathbf{p}}} \quad (7.55)$$



( $\mathcal{S}_t$  can be defined via Cauchy operators as in [16, §12.3]) and

$$\sup_{t \neq 0} \|\nabla_A \mathcal{D}_t f\|_{\mathbf{X}^{\mathbf{p}}} + \|\mathcal{D}_t f\|_{\mathbf{X}^{\mathbf{p}+1}} \lesssim \|f\|_{\mathbf{X}^{\mathbf{p}+1}}. \quad (7.56)$$

We also obtain limits for these operators as  $t \rightarrow 0^\pm$  (in  $\mathbf{X}^{\mathbf{p}}$  or  $\mathbf{X}^{\mathbf{p}+1}$  accordingly, and in the strong or the weak-star topology depending on whether  $\mathbf{p}$  is finite). In particular we can also recover the jump relations with this formalism. We refer the reader to Auscher and Stahlhut [16, §12.3] for further details.

For  $\mathbf{p}$  as above, Rosén's identification of the layer potentials in terms of Cauchy operators and the boundedness results above imply that the solutions to boundary value problems that we construct via Cauchy operators coincide with solutions constructed by the method of layer potentials. It is possible that this fails outside this range of  $\mathbf{p}$ .

This paper is already too long, so details will be left as a challenge to the reader.

-Alan McIntosh, *Operators which have an  $H_\infty$  functional calculus* [\[70\]](#)

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