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# Primal and Dual Approaches to the Analysis of Risk Aversion

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## PRIMAL AND DUAL APPROACHES TO THE ANALYSIS OF RISK AVERSION

In a classic paper, Peleg and Yaari (1975) observe that, for risk-neutral decisionmakers, the marginal rate of substitution between state-contingent income claims is given by the relative probabilities. Thus, probabilities play the same role in problems of choice under uncertainty as do prices in the traditional producer and consumer choice problems. More generally, for risk-averse individuals, the marginal rates of substitution between state-contingent incomes may be interpreted as relative 'risk-neutral' probabilities (Nau, 2001). This analogy is particularly apt when there are spanning portfolios. In that case, after suitable normalization, the equilibrium vector of relative prices for state-contingent claims can be interpreted as a vector of risk-neutral probabilities common to all individuals participating in the market.

These observations suggest that preferences under uncertainty can be analyzed in much the same fashion that one analyzes consumer preferences under certainty, that is, in terms of convex indifference sets and their supporting hyperplanes. This is not a new observation (Yaari, 1969; Lewbel, 1991; Milne, 1995), but, to our knowledge, it has yet to be systematically exploited. The aim of this paper is to extend Peleg and Yaari's fundamental insight by analyzing decisionmaker preferences under uncertainty in terms of convex sets, with probabilities playing a role analogous to that of prices in consumer and producer theory. A key consequence of this observation is the recognition that much of the vast literature on functional structure for consumer preferences and producer technologies can be imported, with proper modification, into the analysis of preferences under uncertainty.

While the analysis of convex indifference sets has proved valuable in consumer and producer theory and other areas, little use seems to have been made of these methods in problems involving uncertainty. Given Arrow (1953) and Debreu's (1952) early demonstrations that problems involving uncertainty are formally identical to those under certainty once the concept of state-contingent commodities is invoked, this seems surprising. We speculate that there are a number of historical reasons.

First, despite the early contributions of Arrow (1953) and Debreu (1952) and the later contributions of Hirshleifer (1965), Yaari (1969), and Peleg and Yaari (1975), the state-contingent approach has been neglected in favor of what Hart and Holmström (1987) refer to as the 'parametrized distribution approach'. That approach focuses attention on families of probability distributions (usually infinite dimensional) over an outcome space, indexed by one or more parameters. By its nature, this characterization does not lend itself to an approach which, as typically applied, involves finite-dimensional price and quantity vectors.

Second, in a complete Arrow-Debreu equilibrium, with a finite state space, the relevant prices are the (unique) prices of state-contingent claims. More generally, even if markets are not complete, the relevant state-claim prices (not necessarily unique) can be derived via Ross's (1976) arbitrage pricing theorem under appropriate conditions. However, in many economic problems involving uncertainty, either there are no financial assets, or only a small number appear to exist relative to the dimension of the state space.

Finally, in areas where the state-contingent approach has been used extensively, including general equilibrium and finance theory (Milne 1995), the primary concern is often with questions of existence, such as the determination of asset prices, and not with welfare evaluations or comparative-static analysis, where the interplay between primal and dual measures can perhpaps be best exploited.

We start by representing preferences in terms of the benefit function, originally developed in the theories of inequality measurement and consumer preferences under certainty (Blackorby and Donaldson, 1980; Luenberger, 1992), and its concave conjugate, which we refer to as the expected-value function. Frequently, problems which prove intractable in the primal representation admit simple solutions in the conjugate representation, and vice versa. A useful example of this is offered by the case of generalized linear risk tolerance, which is easily represented in terms of the conjugate, but for which no closed-form certainty equivalent exists. The second crucial tool in the analysis is the use of superdifferentials that yield simple representations of probabilities as closed, convex sets of relative prices in the spirit of Peleg and Yaari (1975), even when preferences are not smooth.

Next, we consider the notion of risk aversion, beginning with Yaari's (1969) concept of risk aversion as a quasi-concavity property, then defining risk aversion with respect to a probability vector. This definition gives rise to dual versions of the Pratt–Arrow absolute and relative risk premiums as functions of the probabilities. If the individual is risk-averse with respect to a given probability vector, these dual risk premiums take the maximum values (zero and one) at that vector, just as the corresponding primal measures are minimized at certainty.

The power of these tools is illustrated by an analysis of the concepts of constant absolute and relative risk aversion as homotheticity properties. Homotheticity conditions of various kinds play a central role in consumer and producer theory and in their application. In particular, aggregation of consumers and producers, the computation of exact index numbers (including inequality indexes), exact welfare comparisons, and the empirical modelling of consumer demand are facilitated by the existence of appropriately homothetic preferences (Shephard, 1953; Malmquist, 1953; Gorman, 1953, 1981; Diewert, 1976a, 1976b; Muellbauer, 1976; Deaton and Muellbauer, 1980; Caves, Christensen, and Diewert, 1982a, 1982b; Diewert, 1992; Balk, 2000; Chambers, 2001). In this paper, we exploit the observation of Chambers and Färe (1998) and Quiggin and Chambers (1998) that constant absolute risk aversion corresponds to an appropriately-defined notion of translation-homotheticity, and that constant relative risk aversion corresponds to homotheticity. Just as in expected-utility theory, a notion of linear risk tolerance can then be specified which generalizes both of these concepts. Linear risk tolerance, in fact, corresponds to quasi-homotheticity of preferences.

The combination of constant absolute risk aversion and constant relative risk aversion yields constant risk aversion (Safra and Segal 1998). Safra and Segal (1998) demonstrate the theoretical and practical importance of constant risk aversion by showing that, in the presence of linear utility, constant risk aversion, combined with critical features of a number of important generalizations of expected utility theory, is sufficient to characterize that theory. For example, betweenness (Chew 1989) and constant risk aversion are sufficient to characterize disappointment theory (Gul 1991) with linear utility. Important examples of preferences displaying constant risk aversion include the linear mean-standard deviation preferences, completely risk-averse preferences, Yaari's (1987) dual theory, and Weymark's (1981) generalized Gini model. We extend these results by showing that maxmin expected value preferences are the only quasi-concave preferences consistent with constant risk aversion. This demonstration leads to two important further observations. The first offers an alternative axiomatic basis for the Gilboa-Schmeidler (1989) maxmin expected utility preference structure. The second is that preferences are consistent with constant risk aversion if and only if they exhibit the 'plunging' behavior Yaari (1987) observed for his dual preferences.

Karni (1985) has emphasized the importance of state-dependent preferences in the analysis of decisionmaking under uncertainty and has offered generalizations of notions familiar from the state-independent expected-utility model for such preferences. State-dependent expected-utility preferences, of course, are a special case of the preferences that we consider. We show that Karni's (1985) notions of a reference set, a generalized risk premium, and autocomparability of preferences are perhaps most conveniently characterized in terms of the expected-value function. We use that characterization to extend Karni's (1985) results in several dimensions.

In generalizing the Pratt–Arrow meaures of risk aversion (Pratt 1964, Arrow 1965) to general preferences, Nau (2001) has recently introduced the concepts of the buying price, the selling price, the marginal price, and a generalized risk premium of a risky asset for differentiably smooth preferences. We generalize and extend his results in several ways. First, we generalize and characterize his measures for general preferences in terms of superdifferentials and directional derivatives. Among other results we show that the buying and the selling price are only equal under constant absolute risk aversion, and in that case we develop an exact and superlative index of the value of a risky asset in the presence of background risk that extends the standard Pratt–Arrow approximations. We also show that an appropriate version of the generalized risk premium of the risky asset is convex in the risky asset. This result extends a basic property of the standard Pratt–Arrow risk premium.

# 1 Notation

We consider preferences over random variables represented as mappings from a state space $\Omega$ to an outcome space $Y \subseteq \Re$. Our focus is on the case where $\Omega$ is a finite set $\{1, ...S\}$, and the space of random variables is $Y^S \subseteq \Re^S$. The unit vector is denoted $\mathbf{1} = (1, 1, ...1)$, and $\mathcal{P} \subset \Re^S_{++}$ denotes the probability simplex. Define $\mathbf{e}_i$ as the $i$-th row of the $S \times S$ identity matrix

$$\mathbf{e}_i = (0, ..., 1, 0, ..., 0).$$

Preferences over state-contingent incomes are given by an ordinal mapping $W : \Re^S \to \Re$. $W$ is continuous, nondecreasing, and quasi-concave in $\mathbf{y}$. Quasi-concavity ensures that the least-as-good sets of the preference mapping

$$V(w) = \{\mathbf{y} : W(\mathbf{y}) \geq w\}$$

are convex, and that the individual is averse to risk in the sense of Yaari (1969). The *benefit function*, $B : \Re \times Y^S \to \Re$, is defined for $\mathbf{g} \in \Re^S_+$ by:

$$\begin{aligned} B(w, \mathbf{y}; \mathbf{g}) &= \bar{B}(V(w); \mathbf{y}; \mathbf{g}) \\ &= \max\{\beta \in \Re : \mathbf{y} - \beta\mathbf{g} \in V(w)\} \end{aligned}$$

if $\mathbf{y} - \beta\mathbf{g} \in V(w)$ for some $\beta$, and $-\infty$ otherwise (Blackorby and Donaldson, 1980; Luenberger, 1992).[1] The properties of $B(w, \mathbf{y}; \mathbf{g})$ are well known (Luenberger, 1992; Chambers, Chung, and Färe, 1996), and are summarized for later use in the following lemma:

[1]When $\mathbf{g} = \mathbf{1}$ the benefit function corresponds to the *translation function* introduced by Blackorby and Donaldson (1980).

**Lemma 1** $B(w, \mathbf{y}; \mathbf{g})$ *satisfies:*

*a)* $B(w, \mathbf{y}; \mathbf{g})$ *is nonincreasing in* $w$ *and nondecreasing and concave in* $\mathbf{y}$*;*

*b)* $B(w, \mathbf{y} + \alpha\mathbf{g}; \mathbf{g}) = B(w, \mathbf{y}; \mathbf{g}) + \alpha$*,* $\alpha \in \Re$ *(the translation property);*

*c)* $B(w, \mathbf{y}; \mathbf{g}) \geq 0 \Leftrightarrow \mathbf{y} \in V(w)$*;*

*d)* $B(w, \mathbf{y}; \mathbf{g})$ *is jointly continuous in* $\mathbf{y}$ *and* $w$ *in the interior of the region* $\Re \times Y^S$ *where* $B(w, \mathbf{y}; \mathbf{g})$ *is finite.*

The benefit function affords a general method for obtaining alternative representations of preferences. For example, the certainty equivalent is the particular case:

$$\begin{aligned} e(\mathbf{y}) &= \min\{c > 0 : W(c\mathbf{1}) \geq W(\mathbf{y})\} \\ &= -B(W(\mathbf{y}), \mathbf{0}, \mathbf{1}). \end{aligned}$$

The certainty equivalent trivially satisfies

$$e(\mu\mathbf{1}) = \mu, \quad \mu \in \Re.$$

The use of certainty equivalents (generalized mean values) as representations of preferences has been discussed by Chew (1982).

We refer to the concave conjugate of the translation function, $B(w, \mathbf{y}; \mathbf{1})$, as the *expected-value function* $E : \mathcal{P} \times \Re \rightarrow \Re$. It is defined by

$$\begin{aligned} E(\boldsymbol{\pi}, w) &= \inf_{\mathbf{y}} \{\boldsymbol{\pi}\mathbf{y} - B(w, \mathbf{y}; \mathbf{1})\} \quad \boldsymbol{\pi} \in \mathcal{P} \\ &= \inf_{\mathbf{y}} \{\boldsymbol{\pi}\mathbf{y} : B(w, \mathbf{y}; \mathbf{1}) \geq 0\} \quad \boldsymbol{\pi} \in \mathcal{P}, \end{aligned}$$

if there exists some $\mathbf{y}$ such that $B(w, \mathbf{y}; \mathbf{1}) \geq 0$, and $-\infty$ otherwise. The second equality follows by noting that for $\boldsymbol{\pi} \in \mathcal{P}$, $\boldsymbol{\pi}\mathbf{y} - B(w, \mathbf{y}; \mathbf{1}) = \boldsymbol{\pi}[\mathbf{y} - B(w, \mathbf{y}; \mathbf{1})\mathbf{1}]$ and that by definition of $B$ and Lemma 1.c, $B(w, \mathbf{y} - B(w, \mathbf{y}; \mathbf{1})\mathbf{1}; \mathbf{1}) \geq 0$.[2] Denote

$$\mathbf{y}(\boldsymbol{\pi}, w) \in \arg\inf_{\mathbf{y}} \{\boldsymbol{\pi}\mathbf{y} : B(w, \mathbf{y}; \mathbf{1}) \geq 0\}.$$

Because $B(w, \mathbf{y}; \mathbf{1})$ is a continuous and nondecreasing proper concave function, $E(\boldsymbol{\pi}, w)$ is concave and nondecreasing on $\mathcal{P}$ and continuous on the interior of the region of $\mathcal{P}$ where it is finite (Rockafellar, 1970). It is also continuous and nondecreasing in $w$ in the region where it is finite. By conjugacy,

$$B(w, \mathbf{y}; \mathbf{1}) = \inf_{\boldsymbol{\pi} \in \mathcal{P}} \{\boldsymbol{\pi}\mathbf{y} - E(\boldsymbol{\pi}, w)\}.$$

[2] The expected-value function, thus, can be thought of as an expenditure function for the certainty equivalent $e(\mathbf{y})$ in terms of normalized state-claim prices.

## 1.1 Risk-neutral probabilities

Normally in economic theory, the admission of nondifferentiability brings little with it apart from extra theoretical rigour and generality. However recent work has shown that nondifferentiability can be particularly important for generalized expected utility models. In those models, a number of important results turn on the distinction between the concepts of second-order risk aversion that characterizes expected utility theory and Fréchet-differentiable preferences (Machina, 1982) and that of first-order risk aversion that characterizes models such as rank-dependent expected utility (Epstein and Zinn 1990, Segal and Spivak 1990). Recently, Machina (2000) has shown that Fréchet-differentiable and expected-utility models cannot exhibit payoff kinks that emerge as a characteristic of an individual's preferences over lotteries while rank-dependent models cannot avoid exhibiting them. Hence, developing a representation of preferences that relies on differentiability necessarily excludes several important classes of preferences from consideration.

Because the translation function and the expected-value function form a conjugate pair, they offer a natural method for defining and generating subjective notions of probability in terms of their superdifferentials and one-sided directional derivatives. This method allows the analysis of both first-order and second-order risk aversion. Moreover, because both the translation function and the expected-value function are cardinal representations of an ordinal preference structure, there is no need to normalize the superdifferentials and the directional derivatives. Representation in terms of $W(\mathbf{y})$ requires such a normalization.[3]

First some notation and definitions. For a proper concave function $f : \Re^S \to \Re$, its *superdifferential* at $\mathbf{x}$ is the closed, convex set:

$$\partial f(\mathbf{x}) = \left\{ \mathbf{v} \in \Re^S : f(\mathbf{x}) + \mathbf{v}(\mathbf{z} - \mathbf{x}) \geq f(\mathbf{z}) \text{ for all } \mathbf{z} \right\}. \tag{1}$$

The elements of $\partial f(\mathbf{x})$ are referred to as *supergradients*. The *one-sided directional derivative* of $f$ in the direction of $\mathbf{z}$ is defined by

$$f'(\mathbf{x}; \mathbf{z}) = \sup_{\lambda > 0} \left\{ \frac{f(\mathbf{x} + \lambda \mathbf{z}) - f(\mathbf{x})}{\lambda} \right\}.$$

By basic results on proper concave functions for $f$ concave (Rockafellar, 1970):

$$f'(\mathbf{x}; \mathbf{z}) = \inf_{\mathbf{v} \in \partial f(\mathbf{x})} \{\mathbf{v}\mathbf{z}\}.$$

[3] The normalization is accomplished by the choice of the reference asset. In most studies, the reference asset has usually been taken to be the safe asset with payoff vector $\mathbf{1}$, a normalization which we adopt. Karni's (1985) state-dependent approach is based on the assumption that the optimal expansion path and the payoff vector $\mathbf{1}$ need not coincide.

Consequently, $f'(\mathbf{x};\mathbf{z})$ is positively linearly homogeneous and concave in $\mathbf{z}$. Moreover,

$$f'(\mathbf{x};\mathbf{z}) \leq -f'(-\mathbf{x};\mathbf{z}).$$

When $f'(\mathbf{x};\mathbf{z}) = -f'(-\mathbf{x};\mathbf{z})$ for all $\mathbf{z}$, we say that $f$ is differentiable at $\mathbf{x}$. Moreover, if $f$ is differentiable at $\mathbf{x}$, $\partial f(\mathbf{x})$ is a singleton and corresponds to the usual gradient. If $\partial f(\mathbf{x})$ is a singleton, $f$ is differentiable at $\mathbf{x}$ (Rockafellar, 1970).

Because we concern ourselves with ordinal preferences over state-contingent incomes, there is no loss of generality in operating entirely in terms of certainty equivalents. Yaari (1969) identifies subjective probabilities with the supporting hyperplane to $V(e)$ along the sure-thing vector (the bi-sector in two space). These subjective probabilities are given by $\partial B(e, e\mathbf{1}, \mathbf{1})$. Following Nau (2001), it is also informative to consider supporting hyperplanes for the indifference set away from the sure-thing vector.

The translation property of the translation function (Lemma 1(b)) ensures that the superdifferential of $B$ is an element of the unit simplex. In particular:

**Lemma 2** *Let*

$$\mathbf{p}(e,\mathbf{y}) \in \partial B(e,\mathbf{y},\mathbf{1}).$$

*Then*

$$\sum_{s\in\Omega} p_s(e,\mathbf{y}) = 1,$$

*and*

$$\mathbf{p}(e,\mathbf{y}+\delta\mathbf{1}) = p(e,\mathbf{y}), \quad \delta \in \Re.$$

**Proof** By Lemma 1.b,

$$B(e,\mathbf{y}+\delta\mathbf{1},\mathbf{1}) = B(e,\mathbf{y},\mathbf{1}) + \delta$$

If $\mathbf{v} \in \partial B(e,\mathbf{y},\mathbf{1})$ and $\mathbf{z} = \mathbf{y}+\delta\mathbf{1}$, $\mathbf{z}^* = \mathbf{y}-\delta\mathbf{1}$, then

$$\begin{aligned} B(e,\mathbf{y},\mathbf{1}) + \mathbf{v}(\mathbf{z}-\mathbf{y}) &\geq B(e,\mathbf{z},\mathbf{1}) \\ B(e,\mathbf{y},\mathbf{1}) + \mathbf{v}(\mathbf{z}^*-\mathbf{y}) &\geq B(e,\mathbf{z}^*,\mathbf{1}) \end{aligned}$$

or

$$\begin{aligned} B(e,\mathbf{y},\mathbf{1}) + \delta\mathbf{v}\mathbf{1} &\geq B(e,\mathbf{y},\mathbf{1}) + \delta \\ B(e,\mathbf{y},\mathbf{1}) - \delta\mathbf{v}\mathbf{1} &\geq B(e,\mathbf{y},\mathbf{1}) - \delta \end{aligned}$$

so that

$$\sum_{s\in\Omega} v_s = 1.$$

For the second part,

$$\begin{aligned}
\partial B\left(e, \mathbf{y}+\boldsymbol{\delta}\mathbf{1}, \mathbf{1}\right) &= \left\{\mathbf{v} : B\left(e, \mathbf{y}+\boldsymbol{\delta}\mathbf{1}, \mathbf{1}\right) + \mathbf{v}\left(\mathbf{z}+\boldsymbol{\delta}\mathbf{1} - \left[\mathbf{y} + \boldsymbol{\delta}\mathbf{1}\right]\right) \geq B\left(e, \mathbf{z}+\boldsymbol{\delta}\mathbf{1}, \mathbf{1}\right) \text{ for all } \mathbf{z}+\boldsymbol{\delta}\mathbf{1}\right\} \\
&= \left\{\mathbf{v} : B\left(e, \mathbf{y}, \mathbf{1}\right) + \mathbf{v}\left(\mathbf{z} - \mathbf{y}\right) \geq B\left(e, \mathbf{z}, \mathbf{1}\right) \text{ for all } \mathbf{z}\right\} \\
&= \partial B\left(e, \mathbf{y}, \mathbf{1}\right),
\end{aligned}$$

where the second equality follows by Lemma 1.b.■

In view of Lemma 2, the elements of the vector $\mathbf{p}\left(e, \mathbf{y}\right) \subset \Re_{+}^{S}$ are referred to as $e$ *-dependent risk-neutral probabilities*. If the translation function is differentiable, these probabilities are unique and given by the usual gradient. Following Nau (2001), for general risk-averse preferences, we define the set of *risk-neutral probabilities* $\boldsymbol{\pi}\left(\mathbf{y}\right) \subset \Re_{+}^{S}$, which correspond to the supporting hyperplanes for the indifference set, by

$$\begin{aligned}
\boldsymbol{\pi}\left(\mathbf{y}\right) &= \partial B\left(e\left(\mathbf{y}\right), \mathbf{y}; \mathbf{1}\right) \\
&= \mathbf{p}\left(e\left(\mathbf{y}\right), \mathbf{y}\right).
\end{aligned}$$

When preferences are differentiable, $\pi\left(\mathbf{y}\right)$ is a singleton.

By the conjugacy of the translation function and the expected-value function (Rockafellar, 1970),

$$\boldsymbol{\pi} \in \partial B\left(e, \mathbf{y}; \mathbf{1}\right) \Longleftrightarrow \mathbf{y} \in \partial E\left(\boldsymbol{\pi}, e\right) \tag{2}$$

in the relative interior of their domains. Expression (2) is the generalization of Shephard's Lemma to potentially nondifferentiable structures. Thus, the expected-value function can be used as a dual means of obtaining virtual probabilities which correspond to

$$\mathbf{p}\left(e, \mathbf{y}\right) = \arg \inf_{\boldsymbol{\pi}\in\mathcal{P}} \left\{\boldsymbol{\pi}\mathbf{y} - E\left(\boldsymbol{\pi}, e\right)\right\}.$$

The risk-neutral probabilities, therefore, have a natural price interpretation. If the individual can purchase state-claims at relative prices given by $\boldsymbol{\pi} \in \boldsymbol{\pi}\left(\mathbf{y}\right)$, then $\mathbf{y}$ minimises the cost of obtaining the utility level $e\left(\mathbf{y}\right)$. Thus, $\boldsymbol{\pi}\left(\mathbf{y}\right)$ is analogous to an inverse demand correspondence for the Arrow commodities. The risk-neutral probabilities derived here are the preference counterpart to the shadow probabilities considered by Peleg and Yaari (1975), who consider, for a given choice set $C$, the probabilities that would lead a risk-neutral decision-maker to choose $\mathbf{y}$ as the optimal element of $C$.

Following Yaari (1969), the risk-neutral probabilities associated with outcomes along the bisector are of particular interest. Because $e(e\mathbf{1}) = e$, $E(\boldsymbol{\pi},e) \leq e$. And, because preferences are quasi-concave,

$$\boldsymbol{\pi} \in \partial B(e, e\mathbf{1}; \mathbf{1}) \Longleftrightarrow E(\boldsymbol{\pi},e) = e.$$

We, thus, define the set of *subjective probabilities* $\boldsymbol{\pi}(\mathbf{1}) \subset \Re_{+}^{S}$ as

$$\boldsymbol{\pi}(\mathbf{1}) = \cap_e \{\partial B(e, e\mathbf{1}; \mathbf{1})\}.$$

Typically, we shall assume that this set is non-empty, although in general it need not be. In the case of smooth preferences, nonemptiness implies that indifference surfaces are parallel along the sure-thing vector (a form of ray homotheticity). The set will be empty, however, if there is any systematic tendency for the indifference surfaces to 'tilt' as one moves out the sure-thing vector. It is easy to see that this might happen in the case of state-dependent preferences, where the individual's preference for income in any one state relative to others changes systematically as wealth grows. Thus, the set of subjective probabilities satisfies

$$\boldsymbol{\pi}(\mathbf{1}) = \cap_e \arg\sup_{\boldsymbol{\pi}\in\mathcal{P}} \{E(\boldsymbol{\pi},e) - e\}.$$

For an expected utility maximizer with subjective probabilities $\boldsymbol{\pi}$,

$$\{\boldsymbol{\pi}\} = \partial B(e, e\mathbf{1}; \mathbf{1}) \quad \forall e.$$

The elements of $\boldsymbol{\pi}(\mathbf{1})$ will share some, but not all, of the properties of subjective probabilities in expected-utility theory, and more generally, of the probabilities associated with probabilistically sophisticated beliefs in the sense of Machina and Schmeidler (1992). Savage (1954) presents a set of axioms which imply both the existence of well-defined subjective probabilities and an expected-utility representation. Machina and Schmeidler (1992) drop Savage's sure-thing principle, and strengthen Savage's requirements for the consistency of comparative probability judgements to derive conditions under which preferences will preserve first-order stochastic dominance with respect to a unique probability distribution $\boldsymbol{\pi}$. Since the definition of $W$ employed here does not require satisfaction of the sure-thing principle, we will focus on the more general Machina–Schmeidler concept.

The features shared by the superdifferential $\boldsymbol{\pi}(\mathbf{1})$ and the Savage–Machina–Schmeidler concept of subjective probability relate to acceptable betting odds for a risk-averse decision-maker. The definition of the superdifferential of a concave function implies that, beginning with a non-stochastic income $e\mathbf{1}$, welfare will never be improved by acceptance of a bet $\boldsymbol{\varepsilon} \in \Re^S$ which is fair with respect to probabilities $\pi \in \pi(\mathbf{1})$, in the sense that $\boldsymbol{\pi\varepsilon} = 0$. Conversely, if $\pi \notin \pi(\mathbf{1})$, there exists some $e$ and $\boldsymbol{\varepsilon}$ with $\boldsymbol{\pi\varepsilon} = 0$ such that $W(e\mathbf{1} + \boldsymbol{\varepsilon}) > W(e\mathbf{1})$. Moreover, for individuals who are risk-averse and probabilistically sophisticated with subjective probabilities $\boldsymbol{\pi}$ in the sense of Machina and Schmeidler (1992), it must be true that $\pi \in \pi(\mathbf{1})$. If, in addition, preferences are smooth, $\pi(\mathbf{1})$ is a singleton containing the unique probability vector $\boldsymbol{\pi}$ for which an individual with non-stochastic initial income $e\mathbf{1}$ will reject all fair and unfavorable bets (those with $\boldsymbol{\pi\varepsilon} \leq 0$), but will accept all sufficiently small favorable bets. To state the latter condition more precisely, for any $e$ and any $\boldsymbol{\varepsilon}$ with $\boldsymbol{\pi\varepsilon} > 0$, there exists $k > 0$ such that $W(e\mathbf{1}+k\boldsymbol{\varepsilon}) > W(e\mathbf{1})$. Thus, in the case when $W$ is smooth in a neighborhood of the vector $\{e\mathbf{1} : e \in \Re\}$, the unique element $\boldsymbol{\pi} \in \boldsymbol{\pi}(\mathbf{1})$ defines the acceptable betting odds.

The differences between the superdifferential, $\boldsymbol{\pi}(\mathbf{1})$, and the Machina–Schmeidler definition reflect the fact that the superdifferential is a local characterization of preferences for a decision-maker who is assumed to be risk-averse. By contrast, the Machina–Schmeidler definition yields a global stochastic dominance ordering, and the decision-maker is not necessarily risk averse.

The Machina–Schmeidler definition of probabilistic sophistication implies that subjective probabilities are unique, but when preferences are not smooth, in a neighborhood of the vector $\{e\mathbf{1} : e \in \Re\}$, $\boldsymbol{\pi}(\mathbf{1})$ will have more than one element. Consider for example, the case when $S = 2$ and the individual has risk-averse rank-dependent expected utility preferences which preserve first-order stochastic dominance with respect to the unique probability vector $\pi = (\frac{1}{2}, \frac{1}{2})$. The general form of preferences is:

$$W(y_1, y_2) = \begin{cases} q\left(\frac{1}{2}\right) u(y_1) + \left(1 - q\left(\frac{1}{2}\right)\right) u(y_2) & y_1 \leq y_2 \\ q\left(\frac{1}{2}\right) u(y_2) + \left(1 - q\left(\frac{1}{2}\right)\right) u(y_1) & y_2 \leq y_1 \end{cases},$$

where $q$ is the probability weighting function and $u$ is the utility function as in Quiggin (1993). Preferences are risk-averse if $u$ is concave and $q\left(\frac{1}{2}\right) \geq \left(1 - q\left(\frac{1}{2}\right)\right)$.[4]

Now consider bets $(a, b)$ with payoff $a > 0$ received in one state and $-b < 0$ received in the other, and suppose that the individual is free to accept or reject the bet $(ka, kb)$ for any $k > 0$ at

[4]These conditions are sufficient, but not necessary, for the individual to reject all fair bets (Cohen, Chateauneuf and Meilijson, 1997). Sufficiency is all that is required for this illustrative example.

initial wealth $e\mathbf{1}$. For small $k$, the change in welfare associated with increasing the level of the bet is

$$\frac{\partial W}{\partial k} \approx u'(e)\left[-bq\left(\frac{1}{2}\right) + a\left(1 - q\left(\frac{1}{2}\right)\right)\right]$$

which is negative if

$$\frac{a}{b} \leq \frac{q\left(\frac{1}{2}\right)}{\left(1 - q\left(\frac{1}{2}\right)\right)}.$$

Hence,

$$\pi(\mathbf{1}) = \left\{(\pi, 1-\pi) : \left(1 - q\left(\frac{1}{2}\right)\right) \leq \pi \leq q\left(\frac{1}{2}\right)\right\}$$

A second, and more fundamental distinction between the existence of a set of *subjective probabilities* $\boldsymbol{\pi}(\mathbf{1})$, as defined here, and probabilistic sophistication in the sense of Machina and Schmeidler (1992) is that the characterization of $\boldsymbol{\pi}(\mathbf{1})$ depends solely on preferences in a neighborhood of the vector $\{e\mathbf{1} : e \in \Re\}$. Consider an individual with smooth preferences facing an Ellsberg urn problem, with balls of three colours (say 30 red, and 60 either green or yellow). Such an individual might accept all sufficiently small favorable bets at probabilities $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, but might display a preference for bets on the unambiguous outcome red when the payoffs are large. In this case, $\pi(\mathbf{1}) = \left\{\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right\}$, but the individual is not probabilistically sophisticated.

Finally, it should be observed, following Grant and Karni (2000), that if preferences are state-dependent, it is not generally possible to identify subjective probabilities from behavioral observations alone. The identification of $\boldsymbol{\pi}(\mathbf{1})$ with subjective probabilities rests implicitly on an assumption that preferences in the neighborhood of $e\mathbf{1}$ are state-independent.

In summary, our usage of the terms 'risk-neutral probabilities' and 'subjective probabilities' has been adopted to maximize consistency with the literature and for mnemonic simplicity. However, we are concerned solely with the formal properties of probability vectors as elements of the superdifferential. Nothing in the analysis that follows depends on whether the elements of $\boldsymbol{\pi}(\mathbf{y})$ are 'really' probabilities.

## 2 Risk aversion

Early writers on the expected-utility model, such as Friedman and Savage (1948), noted that, in this model, risk aversion was equivalent to concavity of the utility function. The classic work of Pratt (1964) and Arrow(1965) introduced and integrated two approaches to the analysis of risk aversion. The first was a behavioral approach, based on the concept of the risk premium which, in the simplest case, is the difference (or ratio) between the expected value of a risky prospect and the certainty equivalent of that prospect. The second was an index number approach in which risk aversion was characterized by coefficients of absolute (and relative) risk aversion derived from the first and second derivatives of utility functions at a given value $\mathbf{y}$. Arrow and Pratt related the two approaches in a number of ways. First, they showed that an individual would have non-negative risk premiums for all risky prospects if and only if the coefficients of risk aversion were positive for all $\mathbf{y}$. These properties in turn were equivalent to concavity of the utility function. Second, they used the coefficients of risk aversion to derive approximations to the risk premium for prospects in a neighborhood of $\mathbf{y}$. Finally, they characterized the property of constant absolute (relative) risk aversion both behaviorally, by the requirement that a change in base wealth should not change the absolute (relative) risk premium and, in index number terms, by the requirement that the coefficient of absolute (relative) risk aversion should be the same for all $\mathbf{y}$.

The Pratt–Arrow index-number approach has been extended to generalized expected utility models through consideration of local utility functions (Machina 1982), conditions on probability transformations in rank-dependent models (Chew, Karni and Safra 1987) and matrix analogs of the Pratt–Arrow coefficients, applicable to state-dependent utility models (Nau, 2001). Two issues arise here.

First, in these generalized models, the Pratt–Arrow results must be modified. As Machina (1984) observes, the most natural notion of risk aversion for models with smooth preferences, namely that all local utility functions should be concave, is stronger than the requirement for a positive risk premium.

Second, aversion to risk need not imply a preference for outcomes of the form $k\mathbf{1}$ among the set of state-contingent income vectors with a given expectation at the known objective probabilities. As Nau (2001) observes, risk-averse preferences may be characterized by state-dependent utility or by the existence of undiversifiable background risk, such that, if objective probabilities are given by $\pi$, it need not be true that $\pi \in \pi(\mathbf{1})$.

In this paper, we focus on the most basic concept of risk aversion. Following Yaari (1969), risk aversion may be regarded as the property that preferences are convex over the state-contingent outcome space $Y^S$ or, equivalently, that $W$ satisfies the quasi-concavity assumption imposed above.

We first consider the case examined in the majority of the literature on primal measures of risk aversion, where nonstochastic outcomes of the form $k\mathbf{1}$ are preferred to stochastic outcomes with the same expectation. That is, we assume the existence of a probability vector $\boldsymbol{\pi}^0$ such that for all

$e, \mathbf{y}$ with $e(\mathbf{y}) = e$, $\boldsymbol{\pi}^0\mathbf{y} \geq e$ with a corresponding risk premium

$$\boldsymbol{\pi}^0\mathbf{y} - e(\mathbf{y}) \geq 0.$$

Dually, we can define:

**Definition 3** *An individual is risk-averse with respect to probabilities* $\boldsymbol{\pi}^0$ *if* $\boldsymbol{\pi}^0 \in \pi(\mathbf{1})$.

The definition implies that an individual is risk-averse with respect to $\boldsymbol{\pi}^0$ if, from an initial position of certainty represented by some $e\mathbf{1}$, the individual will reject all bets $\mathbf{z}$ that are fair in the sense that $\boldsymbol{\pi}^0\mathbf{z} = 0$ and, *a fortiori*, all bets that are unfavorable in the sense that $\boldsymbol{\pi}^0\mathbf{z} < 0$. In the case where $\boldsymbol{\pi}(\mathbf{1})$ is empty, there exists no probability vector with this property for all $e$. However, it may still be possible to interpret preferences as risk-averse with respect to some set of state-dependent preferences (Grant and Karni, 2000).

The preceding discussion implies:

**Lemma 4** *An individual is risk-averse with respect to probabilities* $\pi^0$ *if and only if:*

$$E\left(\boldsymbol{\pi}^0, e\right) = e \quad \forall e.$$

Consider the two polar cases of nondecreasing, quasi-concave preferences that are risk-averse with respect to a given set of probabilities, $\boldsymbol{\pi}^0$ . They are risk-neutral preferences,

$$e^{\boldsymbol{\pi}^0 N}(\mathbf{y}) = \sum_{s \in \Omega} \pi_s^0 y_s$$

and maximin preferences (corresponding to complete aversion to risk)

$$e^M(\mathbf{y}) = \min\{y_1, y_2, ..., y_S\}.$$

For any other members of the class of nondecreasing, quasi-concave preferences that are risk averse with respect to $\boldsymbol{\pi}^0$, with corresponding at-least-as-good sets $V^{\boldsymbol{\pi}^0}$, it is true that

$$V^M(e) \subseteq V^{\boldsymbol{\pi}^0}(e) \subseteq V^{\boldsymbol{\pi}^0 N}(e).$$

Our general definition of an increase in risk aversion between individuals reflects this basic property that an increase in risk aversion is reflected by a shrinking of the at-least-as-good sets.

**Definition 5** *A is more risk-averse than B if for all e*

$$V^A(e) \subseteq V^B(e).$$

There are several immediate consequences of these definitions. We summarize them in the following theorem and corollary:

**Theorem 6** *The following are equivalent:*

*(a) A is more risk averse than B;*

*(b)* $B^A(e, \mathbf{y}; \mathbf{1}) \leq B^B(e, \mathbf{y}; \mathbf{1})$ *for all* $\mathbf{y}$ *and* $e$*;*

*(c)* $E^A(\boldsymbol{\pi}, e) \geq E^B(\boldsymbol{\pi}, e)$ *for all* $\pi$ *and* $e$*;and*

*(d) for all* $\mathbf{y}$, $e^A(\mathbf{y}) \leq e^B(\mathbf{y})$.

*Moreover, if A is more risk-averse than B, and B is risk-averse with respect to probabilities* $\boldsymbol{\pi}^0$*, so is A.*

**Proof** *(a)⇒(c) is immediate. (c)⇒(b) follows by applying* $E^A(\boldsymbol{\pi}, e) \geq E^B(\boldsymbol{\pi}, e)$ *for all* $\pi$ *and* $e$ *in the conjugacy mapping. (b)⇒(d) follows because* $e(\mathbf{y})$ *is determined by*

$$\max\{e : B(e, \mathbf{y}; \mathbf{1}) \geq 0\}.$$

*(d)⇒(a) is immediate from the definition of* $V$*. The second part of the theorem is trivial.*■

***Corollary 7*** *If both A and B are expected utility maximizers then A is more risk-averse than B if and only if* $u^A$ *is a concave transformation of* $u^B$*.*

Corollary 7 follows from Arrow (1965) and Pratt (1964), who show that A will have lower certainty equivalents than B if and only if $u^A$ is a concave transformation of $u^B$. As part (d) of Theorem 6 shows, this is equivalent to our definition of more risk averse.

## 2.1 Dual Measures of risk aversion

We consider an absolute and a relative measure of risk aversion. The *dual absolute risk premium* is defined as

$$a(\boldsymbol{\pi}, e) = E(\boldsymbol{\pi}, e) - e,$$

and the *dual relative risk premium* (defined only for $e > 0$) as

$$r(\boldsymbol{\pi}, e) = \frac{E(\boldsymbol{\pi}, e)}{e}.$$

Notice that $a(\boldsymbol{\pi}, e) \leq 0$ and $r(\boldsymbol{\pi}, e) \leq 1$. Moreover, because $E$ is concave in $\boldsymbol{\pi}$, so are $a$ and $r$. These two measures are directly related in the case $e > 0$ by

$$a(\boldsymbol{\pi}, e) = e(r(\boldsymbol{\pi}, e) - 1).$$

We observe:

**Lemma 8** *The following conditions are equivalent:*

*(1) A is more risk-averse than B;*

*(2)* $a^A(\boldsymbol{\pi},e) \geq a^B(\boldsymbol{\pi},e) \quad \forall \pi, e$*; and*

*(3) for all* $e > 0$ $r^A(\boldsymbol{\pi},e) \geq r^B(\boldsymbol{\pi},e) \quad \forall \pi$.

**Lemma 9** *An individual is risk-averse with respect to probabilities* $\boldsymbol{\pi}^0$ *if and only if* $a(\boldsymbol{\pi}^0,e) = 0$ *and* $r(\boldsymbol{\pi}^0,e) = 1$.

The polar cases of risk neutrality and complete aversion to risk illustrate the properties of these two measures. If preferences are risk-neutral with respect to $\pi^0$:

$$\begin{aligned}
E(\boldsymbol{\pi},e) &= \inf_{\mathbf{y}} \left\{ \boldsymbol{\pi}\mathbf{y} - \sum_s \pi_s^0 y_s + e \right\} \\
&= e + \inf_{\mathbf{y}} \left\{ \boldsymbol{\pi}\mathbf{y} - \sum_s \pi_s^0 y_s \right\} \\
&= \begin{cases} -\infty & \boldsymbol{\pi} \neq \boldsymbol{\pi}^0 \\ e & \boldsymbol{\pi} = \boldsymbol{\pi}^0 \end{cases}.
\end{aligned}$$

Thus,

$$a(\boldsymbol{\pi},e) = \begin{cases} -\infty & \boldsymbol{\pi} \neq \boldsymbol{\pi}^0 \\ 0 & \boldsymbol{\pi} = \boldsymbol{\pi}^0 \end{cases},$$

and

$$r(\boldsymbol{\pi},e) = \begin{cases} -\infty & \boldsymbol{\pi} \neq \boldsymbol{\pi}^0 \\ 1 & \boldsymbol{\pi} = \boldsymbol{\pi}^0 \end{cases}.$$

It follows immediately that:

**Lemma 10** *An individual is risk-averse with respect to probabilities* $\pi^0$ *if and only if he is more risk-averse than an individual with preferences that are risk-neutral with respect to* $\pi^0$.

For completely risk averse preferences preferences,

$$e(\mathbf{y}) = \min\{y_1, y_2, ..., y_S\},$$

whence

$$\begin{aligned}
E(\boldsymbol{\pi},e) &= \min_{\mathbf{y}} \{\boldsymbol{\pi}\mathbf{y} - \min\{y_1, y_2, ..., y_S\}\} + e \\
&= e
\end{aligned}$$

because $\boldsymbol{\pi}\mathbf{y} - \min\{y_1, y_2, ..., y_S\} \geq 0$. Hence,

$$\begin{aligned} a(\boldsymbol{\pi}, e) &= 0, \\ r(\boldsymbol{\pi}, e) &= 1. \end{aligned}$$

Preferences exhibit *constant absolute risk aversion* if for all $\boldsymbol{\pi}$

$$a(\boldsymbol{\pi}, e) = a(\boldsymbol{\pi}, e') \quad \text{all } e, e'.$$

Preferences exhibit *constant relative risk aversion* if for all $\boldsymbol{\pi}$

$$r(\boldsymbol{\pi}, e) = r(\boldsymbol{\pi}, e') \quad \text{all } e, e' > 0.$$

Our next result shows that these dual notions of constant absolute risk aversion and constant relative risk aversion are equivalent to the more familiar notions. It also characterizes the risk-neutral probabilities for both classes of preferences.

**Theorem 11** *Preferences exhibit constant absolute risk aversion if and only if*

$$E(\boldsymbol{\pi}, e) = \mathring{a}(\boldsymbol{\pi}) + e,$$

*where* $\mathring{a}(\boldsymbol{\pi}) \leq 0$ *is a nondecreasing proper concave function that is continuous on the interior of the region of* $\mathcal{P}$ *where it is finite,*

$$B(e, \mathbf{y}; \mathbf{1}) = B(0, \mathbf{y}; \mathbf{1}) - e,$$

*and*

$$\boldsymbol{\pi}(\mathbf{y} + \beta\mathbf{1}) = \boldsymbol{\pi}(\mathbf{y}), \quad \beta \in \Re.$$

*Preferences exhibit constant relative risk aversion if and only if*

$$E(\boldsymbol{\pi}, e) = \mathring{r}(\boldsymbol{\pi})\, e$$

*where* $\mathring{r}(\boldsymbol{\pi}) \leq 1$ *is a proper concave function that is continuous on the region of* $\mathcal{P}$ *where it is finite,*

$$B(e, \mathbf{y}; \mathbf{1}) = eB\left(1, \frac{\mathbf{y}}{e}; 1\right),$$

*and*

$$\boldsymbol{\pi}(\mu\mathbf{y}) = \boldsymbol{\pi}(\mathbf{y}), \quad \mu > 0.$$

**Proof** *The proof is for CARA. The proof for CRRA is parallel. By CARA*

$$a(\boldsymbol{\pi},e) = \hat{a}(\boldsymbol{\pi}),$$

*with* $\hat{a}(\boldsymbol{\pi}) \leq 0$ *a nondecreasing proper concave function that is continuous on the region of* $\mathcal{P}$ *where it is finite by the properties of the expected-value function. Hence,*

$$E(\boldsymbol{\pi},e) = \hat{a}(\boldsymbol{\pi}) + e.$$

*By conjugacy,*

$$\begin{aligned} B(e,\mathbf{y};\mathbf{1}) &= \min_{\boldsymbol{\pi}} \{\boldsymbol{\pi}\mathbf{y} - \hat{a}(\boldsymbol{\pi})\} - e \\ &= B(0,\mathbf{y};\mathbf{1}) - e, \end{aligned}$$

*where* $B(0,\mathbf{y};\mathbf{1})$ *is the concave conjugate of* $\hat{a}(\boldsymbol{\pi})$ *satisfying the properties in Lemma 1. Because*

$$B(e,\mathbf{y};\mathbf{1}) = B(0,\mathbf{y};\mathbf{1}) - e,$$

*it follows that*

$$\mathbf{p}(e,\mathbf{y}) = \mathbf{p}(0,\mathbf{y})$$

*for all* $\mathbf{y}$*. By the second part of Lemma 2,*

$$\mathbf{p}(0,\mathbf{y}+\beta\mathbf{1}) = \partial B(0,\mathbf{y}+\beta\mathbf{1};\mathbf{1}) = \partial B(0,\mathbf{y};\mathbf{1}) = \mathbf{p}(0,\mathbf{y}).\blacksquare$$

***Corollary 12*** *If preferences exhibit constant absolute risk aversion*

$$\boldsymbol{\pi} \in \cap_e \{\partial B(e, e\mathbf{1};\mathbf{1})\} \Longleftrightarrow \hat{a}(\boldsymbol{\pi}) = 0.$$

*If preferences exhibit constant relative risk aversion*

$$\boldsymbol{\pi} \in \cap_e \{\partial B(e, e\mathbf{1};\mathbf{1})\} \Longleftrightarrow \hat{r}(\boldsymbol{\pi}) = 1.$$

**Remark 1** *A direct consequence of Lemma 1c and Theorem 11 is that*

$$e(\mathbf{y}) = B(0,\mathbf{y};\mathbf{1})$$

*in the case of constant absolute risk aversion, and thus by Lemma 1.b,*

$$e(\mathbf{y}+\beta\mathbf{1}) = e(\mathbf{y}) + \beta,$$

*which is the more traditional definition of constant absolute risk aversion (Quiggin and Chambers, 1998). Hence, $W(\mathbf{y})$ is translation homothetic (Blackorby and Donaldson, 1980; Chambers and Färe, 1998). Similarly, in the case of constant relative risk aversion $e(\mathbf{y})$ is the implicit solution to*

$$B\left(1, \frac{\mathbf{y}}{e}; 1\right) = 0,$$

*whence*

$$e(\mu\mathbf{y}) = \mu e(\mathbf{y}) \quad \mu > 0,$$

*and $W(\mathbf{y})$ is homothetic.*

Because CRRA corresponds to homotheticity of the welfare functional and CARA corresponds to translation homotheticity of the welfare functional, it is natural to speculate that the class of quasi-homothetic preferences, which contains both CRRA and CARA preferences as subsets, will prove useful. Quasi-homothetic preferences are characterized by the fact that their income-expansion paths are straight lines which do not necessarily emanate from the origin. In the expected-utility literature, this characteristic of linear expansion paths has come to be associated with preferences that exhibit *linear risk tolerance (*Brennan and Kraus, 1976; Milne, 1979). We, therefore, say that preferences exhibit *linear risk tolerance* if

$$E(\boldsymbol{\pi},e) = E^0(\boldsymbol{\pi}) + E^1(\boldsymbol{\pi})\, e$$

where $E^0(\boldsymbol{\pi})$ and $E^1(\boldsymbol{\pi}) \geq 0$ are expected-value functions for least-as-good sets that are independent of the certainty equivalent. CARA is the special case of linear risk tolerance where $E^1(\boldsymbol{\pi}) = \boldsymbol{\pi}\mathbf{1} = 1$ for all $\boldsymbol{\pi}$, while CRRA is the special case of linear risk tolerance where $E^0(\boldsymbol{\pi}) = \boldsymbol{\pi}\mathbf{0} = 0$ for all $\boldsymbol{\pi}$.

The associated risk premiums are given by:

$$\begin{aligned} a(\boldsymbol{\pi},e) &= E^0(\boldsymbol{\pi}) + e\left(E^1(\boldsymbol{\pi}) - 1\right) \\ r(\boldsymbol{\pi},e) &= \frac{E^0(\boldsymbol{\pi})}{e} + E^1(\boldsymbol{\pi}). \end{aligned}$$

Because $e$ can take all positive values, for two individuals with linear risk tolerance, $i$ is more risk-averse than $j$ if and only if

$$\begin{aligned} E_i^0(\boldsymbol{\pi}) &\geq E_j^0(\boldsymbol{\pi}) \\ E_i^1(\boldsymbol{\pi}) &\geq E_j^1(\boldsymbol{\pi}). \end{aligned}$$

Both CRRA and CARA preferences are particularly tractable analytically in either their dual or their primal formulation. Preferences exhibiting linear risk tolerance are much simpler when expressed in terms of the expected-value function. As is well known, quasi-homothetic preferences generally do not have a closed form certainty equivalent. The manifestation of this in terms of the translation function is derived directly from composition rules due to Chambers, Chung, and Färe(1996):

**Lemma 13** *Preferences exhibit linear risk tolerance if and only if*

$$B(e, \mathbf{y}; \mathbf{1}) = \sup\left\{\min\left\{B^0\left(\mathbf{y}^0; \mathbf{1}\right), eB^1\left(\frac{\mathbf{y}^1}{e}; 1\right)\right\} : \mathbf{y}^0 + \mathbf{y}^1 = \mathbf{y}\right\},$$

*where $B^0$ is the translation function conjugate to $E^0$, and $B^1$ is the translation function conjugate to $E^1$.*

Because preferences with linear risk tolerance generally do not have closed form welfare functionals, they have received only limited attention in the literature on primal representation of preferences over stochastic incomes. Indeed, even using the more general concept of a superdifferential, it will generally be difficult, and intuitively uninformative to attempt a primal evaluation of the risk-neutral and subjective probabilities. However, one special case that has received attention because of its convenient ability to represent market outcomes in terms of a representative consumer are affinely homothetic preferences (Milne, 1979), which are the special case of linear risk tolerance given by

$$E^0(\boldsymbol{\pi}) = \boldsymbol{\pi}\mathbf{v} \quad \mathbf{v} \in \Re^S_+.$$

Perhaps the best known member of the affinely homothetic class of preferences is the Stone–Geary utility structure which underlies the linear-expenditure system. The translation function conjugate to the class of affinely homothetic expected-value functions, subject to appropriate domain restrictions, is

$$B(e, \mathbf{y}; \mathbf{1}) = eB^1\left(\frac{\mathbf{y} - \mathbf{v}}{e}; \mathbf{1}\right),$$

where $B^1$ is conjugate to $E^1$.

Another special case of the quasi-homothetic family, one which does not appear to have received much attention in the literature on portfolio-selection and uncertainty, is the class of preferences that are translation homothetic in a direction other than that given by the certainty vector. This class, which has played a role in the empirical modelling of labor demand and consumer preferences

(Blackorby, Boyce, Russell, 1978; Dickinson, 1980) is defined by

$$E^1(\boldsymbol{\pi}) = \boldsymbol{\pi}\mathbf{u},$$

where $\mathbf{u} \in \Re^S$. Intuitively, this is the class of preferences for which real income effects are independent of the economic environment. CARA corresponds to the special case where real income effects are constant and the same for all states of nature, i.e., $\mathbf{u} = \mathbf{1}$. The translation function conjugate to the translation homothetic expected-value function is

$$B(e, \mathbf{y}; \mathbf{1}) = B^0(\mathbf{y} - e\mathbf{u}; \mathbf{1}),$$

where $B^0$ is conjugate to $E^0$.

Preferences satisfying constant relative risk aversion, constant absolute risk aversion, and linear risk-tolerance can all be characterized in terms of the notion of demand rank, which corresponds to the dimension of the function space spanned by the individual's Engel curves in budget-share form (Lewbel, 1991). By our results and Theorem 1 of Lewbel (1991), constant relative risk aversion corresponds to a rank-one demand system, while constant absolute risk aversion, and linear risk-tolerance each correspond to rank-two demand systems. Further, using the general results of Lewbel and Perraudin (1995), this establishes that each of these preference structures satisfy the conditions for portfolio separation associated with the theory of mutual funds. Lewbel and Perraudin (1995) show that a necessary and sufficient condition for portfolio separation, with smooth preferences, is that

$$E(\boldsymbol{\pi}, e) = E'\left(\rho^1(\boldsymbol{\pi}), ..., \rho^K(\boldsymbol{\pi}), e\right)$$

where $K < S$.

Constant relative risk aversion, thus, implies that preferences can be represented indirectly in terms of a single mutual fund, and the corresponding holdings of the respective state-claims per unit of real income are given by the gradient of $\hat{r}(\boldsymbol{\pi})$. Constant absolute risk aversion is associated with preferences that can be represented indirectly in terms of two mutual funds, one of which is degenerate and corresponds to the traditionally safe asset. In the terminology of Cass and Stiglitz (1970), there is monetary separation. Only the holding of the degenerate mutual fund is affected by the level of real wealth, and it is this characteristic of constant absolute risk aversion which yields the well-known result that changes in real wealth do not affect the individual's holding of the risky asset. Linear risk tolerance is the generalization of two-mutual fund preferences which makes neither mutual fund degenerate.

In the literature on expected utility preferences, it is well known that only risk-neutral preferences can jointly exhibit constant absolute risk aversion and constant relative risk aversion. Safra and Segal (1998) have recently investigated these type of preferences, which they refer to as constant risk aversion, in the case of an infinite dimensional state space. Preferences with constant risk aversion are interesting not only because they encompass both the important polar cases of risk neutrality and maximal risk aversion, but also because a number of widely-used representations of risk preferences display this property, including Yaari's (1987) dual model, and preferences linear in the mean and standard deviation. Safra and Segal characterize the requirements for constant risk aversion in a number of models of choice under uncertainty.

Quiggin and Chambers (1998), without imposing quasi-concavity, show that preferences defined over a finite state space exhibit constant risk aversion if and only if

$$B(e, \mathbf{y}; \mathbf{1}) = g(\mathbf{y} - Min\{y_1, ..., y_S\}\mathbf{1}) + Min\{y_1, ..., y_S\} - e,$$

with $g$ positively linearly homogeneous. Maxmin, linear mean-standard deviation, and risk-neutral preferences are all special cases of this preference structure. The expected value function for this class of preferences can be derived as

$$\begin{aligned} E(\boldsymbol{\pi}, e) &= \inf_{\mathbf{y}} \{\boldsymbol{\pi}\mathbf{y} - Min\{y_1, ..., y_S\} - g(\mathbf{y} - Min\{y_1, ..., y_S\}\mathbf{1})\} + e \\ &= \inf_{\mathbf{y}} \{\boldsymbol{\pi}(\mathbf{y} - Min\{y_1, ..., y_S\}\mathbf{1}) - g(\mathbf{y} - Min\{y_1, ..., y_S\}\mathbf{1})\} + e \\ &= \inf_{\hat{\mathbf{y}}} \{\boldsymbol{\pi}\hat{\mathbf{y}} - g(\hat{\mathbf{y}})\} + e. \end{aligned}$$

Because $g$ is positively linearly homogeneous, $\inf_{\hat{\mathbf{y}}} \{\boldsymbol{\pi}\hat{\mathbf{y}} - g(\hat{\mathbf{y}})\}$ equals either 0 or $-\infty$. This observation leads to the following characterization of quasi-concave preferences consistent with constant risk aversion:

**Theorem 14** *Preferences exhibit constant risk aversion if and only if*

$$E(\boldsymbol{\pi}, e) = \begin{cases} e & \boldsymbol{\pi} \in \hat{\mathcal{P}} \\ -\infty & \boldsymbol{\pi} \notin \hat{\mathcal{P}} \end{cases},$$

*and*

$$B(e, \mathbf{y}; \mathbf{1}) = \inf\left\{\boldsymbol{\pi}\mathbf{y} : \boldsymbol{\pi} \in \hat{\mathcal{P}}\right\} - e,$$

*for* $\hat{\mathcal{P}} \subseteq \mathcal{P}$ *closed.*

**Proof** *By Theorem 11, preferences exhibit constant absolute risk aversion if and only if*

$$E(\boldsymbol{\pi},e) = \hat{a}(\boldsymbol{\pi}) + e$$

*where* $\hat{a}(\boldsymbol{\pi}) \leq 0$ *is a proper concave translation function. To satisfy constant relative risk aversion, it, further follows from Theorem 11 that*

$$\mu\hat{a}(\boldsymbol{\pi}) = \hat{a}(\boldsymbol{\pi}) \quad \mu > 0.$$

*There are three possibilities either* $\hat{a}(\boldsymbol{\pi}) = 0$, $\hat{a}(\boldsymbol{\pi}) = \infty$ *or* $\hat{a}(\boldsymbol{\pi}) = -\infty$. *If* $\hat{a}(\boldsymbol{\pi}) = \infty$, *there is no* $\mathbf{y}$ *such that* $B(e,\mathbf{y};\mathbf{1}) \geq 0$, *and hence* $V(e)$ *is empty. If* $\hat{a}(\boldsymbol{\pi}) = -\infty$ *for all* $\boldsymbol{\pi}$, *preferences are not well defined, and that case is ruled out. The only proper, concave function remaining is*

$$\hat{a}(\boldsymbol{\pi}) = \begin{cases} 0 & \boldsymbol{\pi} \in \hat{\mathcal{P}} \\ -\infty & \boldsymbol{\pi} \notin \hat{\mathcal{P}} \end{cases},$$

*for* $\hat{\mathcal{P}} \subseteq \mathcal{P}$ *closed. This establishes necessity of the first part. Sufficiency of the first part follows trivially. By the conjugacy of the translation and expected-value functions:*

$$B(e,\mathbf{y};\mathbf{1}) = \inf_{\boldsymbol{\pi}\in\mathcal{P}} \{\boldsymbol{\pi}\mathbf{y} - E(\boldsymbol{\pi},e)\}.$$

*For all* $\boldsymbol{\pi} \notin \hat{\mathcal{P}}$, $\boldsymbol{\pi}\mathbf{y} - E(\boldsymbol{\pi},e) = \infty$, *and thus*

$$B(e,\mathbf{y};\mathbf{1}) = \inf_{\boldsymbol{\pi}} \left\{\boldsymbol{\pi}\mathbf{y} - E(\boldsymbol{\pi},e) : \boldsymbol{\pi} \in \hat{\mathcal{P}}\right\} < \infty,$$

*if it is to be finite.*■

Quasi-concave preferences are consistent with constant risk aversion if and only if they correspond to the support function for a closed subset of the probability simplex, or perhaps more intuitively, if and only if they belong to the maxmin expected-value (MMEV) class. MMEV preferences are closely related to the maxmin expected utility class (MMEU) introduced by Gilboa and Schmeidler (1989) and extended by Casadesus-Masanell, Klibanoff, and Ozdenoren (2000a, 2000b). The MMEU class is represented by the functional

$$W(\mathbf{y}) = \inf_{\boldsymbol{\pi}} \left\{\boldsymbol{\pi}\mathbf{u}(\mathbf{y}) : \boldsymbol{\pi} \in \hat{\mathcal{P}}\right\}$$

where $u_s(\mathbf{y}) = u(y_s)$ for some concave von Neumann-Morgenstern utility function $u$.

Constant risk averse preferences, therefore, have the alternative interpretation as modelling the behavior of innately risk-neutral individuals who have too little information to form a prior probability distribution, and thus choose by forming minimal expected values over a set of possible prior distributions. At one extreme, their information is so inexact that this minimal expected value is taken over the entire probability simplex

$$\begin{aligned} B(e, \mathbf{y}; \mathbf{1}) &= \inf \{\boldsymbol{\pi}\mathbf{y} : \boldsymbol{\pi} \in \mathcal{P}\} - e \\ &= \min \{y_1, y_2, ..., y_S\} - e. \end{aligned}$$

At the other extreme is the other polar case where $\mathcal{P} = \{\boldsymbol{\pi}^0\}$, and preferences are risk neutral. In between risk neutrality and complete aversion to risk lay the class of preferences characterized by piecewise linear indifference curves. MMEV, just as MMEU, preferences, therefore, exhibit uncertainty aversion in the sense that as the set of priors from which the individual can choose grows (in an inclusion sense), her evaluation of a given $\mathbf{y}$ cannot rise.

A striking implication of this result is that classes of preferences known to display constant risk aversion, including preferences additively separable in the mean and the standard deviation and the generalized Leontief preferences considered by Quiggin and Chambers (1998), must be representable by a preference functional of the MMEV class.

It is extremely well-known that among the class of expected utility preferences, only risk-neutral preferences exhibit constant risk aversion. It is an obvious corollary of Theorem 14 that the only constant risk averse preferences consistent with an expected utility functional corresponds to the case where $\mathcal{P}$ is a singleton. More recently, Chambers and Quiggin (2000) have shown that strictly quasi-concave preferences cannot exhibit constant risk aversion. Strict quasi-concavity of the preference structure is reflected in strict concavity of the benefit function (superdifferentials unique up to a positive scalar multiple), which is not consistent with Theorem 14. It is also obvious that Yaari's (1987) dual linear model is a special case of the quasi-concave constant risk averse preference functional.

**Remark 2** *Theorem 14 suggests an alternative axiomatic basis for the MMEU class of preferences as the class consistent with constant risk aversion in a state-independent transform of $y_s$. Similarly, state-dependent preferences can be made consistent with this form of uncertainty aversion by requiring that preferences satisfy constant risk aversion in state-dependent transforms of $y_s$*

Besides exhaustively characterizing the class of constant risk averse preferences, Theorem 14 demonstrates that these structural restrictions have important behavioral consequences.

**Corollary 15** *Preferences exhibit constant risk aversion if and only if either* $\partial E(\boldsymbol{\pi}, e) = e\mathbf{1}$ *or* $\partial E(\boldsymbol{\pi}, e)$ *is undefined.*

By (2), this corollary generalizes Yaari's (1987) observation that preferences in his dual model display 'plunging' behavior. Either the individual will reject a given risk entirely and adopt a non-stochastic portfolio, or he will accept an amount of the risk that is either unbounded or fixed by the constraints of the choice problem. Corollary 15 establishes the more general result that plunging behavior characterizes the entire class of quasi-concave, constant risk averse preferences.

Dual notions of decreasing risk aversion prove useful in comparative-static analysis. We have:

**Definition 16** *Preferences display decreasing absolute risk aversion if for all* $\pi$, $a(\boldsymbol{\pi},e)$ *is decreasing in* $e$.

**Definition 17** *Preferences display decreasing relative risk aversion if for all* $\boldsymbol{\pi}$, $r(\boldsymbol{\pi},e)$ *is decreasing in* $e > 0$.

The following corollary is a trivial consequence of the definitions and Theorem 11:

**Corollary 18** *If preferences exhibit constant absolute risk aversion, they exhibit nondecreasing relative risk aversion. If preferences exhibit constant relative risk aversion, they exhibit nonincreasing absolute risk aversion.*

## 2.2 Reference Sets and Karni risk premiums

Thus far, we have focused on the case where nonstochastic outcomes of the form $k\mathbf{1}$ are preferred to stochastic outcomes with the same expectation. This will not, in general, be the case if preferences are state-dependent. We close our discussion of dual risk aversion with an illustration of how $E(\boldsymbol{\pi}, e)$ can be used to conveniently characterize existing concepts from the literature on state-dependent preferences. Karni (1985) gives a systematic exposition of the state-dependent expected utility model, parametrized by the probability vector $\hat{\boldsymbol{\pi}}$:

$$W(\mathbf{y}) = \sum_s \hat{\pi}_s u_s(y_s).$$

Grant and Karni (2000) examine state-dependent preferences in the absence of the additive separability assumption implied by an expected-utility formulation, but do not give a full extension of Karni's (1985) results. In this section, we show how the main concepts of Karni (1985) may be developed in dual terms, without reference to the expected utility model.

Karni (1985) defines the *reference set* as "...the optimal distribution of wealth across states of nature that is chosen by a risk-averse decision maker facing fair insurance" at the probabilities $\boldsymbol{\pi}$. Hence, the reference set corresponds in a consumer context to the consumer's income-expansion path given $\boldsymbol{\pi}$. For given $\boldsymbol{\pi}$, the reference set is thus

$$RS(\boldsymbol{\pi}) = \cup_e \{\mathbf{y}(\boldsymbol{\pi}, e)\},$$

and has the equivalent interpretation in our framework as the optimal distribution of "real wealth" across states of nature as chosen by a risk-averse decisionmaker given the normalized state-claim prices $\boldsymbol{\pi}$. Both Karni's (1985) state-dependent model and expected-utility preferences have the convenient property that $RS(\boldsymbol{\pi})$ is independent of $\boldsymbol{\pi}$ provided that $E(\boldsymbol{\pi}, e)$ is evaluated at the $\hat{\boldsymbol{\pi}}$ that parameterizes the preference structure.[5]

More generally, however, risk-averse preferences can have reference sets that are not independent of $\boldsymbol{\pi}$. The independence of the reference set in the state-dependent and state-independent expected utility models is a consequence of the linear-in-probabilities model's ability to separate changes in beliefs about probabilities and changes in preferences over money income. In these models, the independence of $RS(\boldsymbol{\pi})$ reflects the fact that the most preferred set, given the probabilities, is determined by the individual's preferences over money income. Given the probabilities, state-independent expected utility, by definition, does not discriminate among income in different states and, hence, $RS(\boldsymbol{\pi}) = \{\alpha\mathbf{1} : \alpha \in \Re\}$. State-dependent expected utility permits taste-based discrimination among income in different states, rising for example from background risk, while preserving independence of the reference set.

In more general risk-averse models, tastes and probability beliefs are not so neatly separable. However, the notion of a reference set still allows sensible local comparisons across individuals facing either objectively fair probabilities $\pi$ or common exogenous state-claim prices. Two risk-averse individuals having a common reference set for $\boldsymbol{\pi}$ make the same choices over actuarially fair insurance for different income levels. One can always approximate their choices for these supporting state-claim prices locally by state-independent or state-dependent expected utility functionals sharing the same reference set. Karni's notions of more risk averse and less risk averse thus remain locally relevant for general risk-averse preferences. However, the precise ability to attribute different behavior over fair gambles around the reference set to pure differences in attitudes toward risk,

[5] Karni (1985) appears to restrict attention exclusively to the case where in our terminology $E(\square, e)$ is always calculated for the □ that parametrizes preferences.

which expected-utility models have, is not inherited by more general risk-averse models. But the ability to discriminate among differing behaviors on the basis of these measures offers an important point of reference for comparative-static analysis.

While independence is an important characteristic to have, even without it, other commonly understood restrictions on risk-averse preferences lead to particularly convenient reference sets.

**Theorem 19** *Preferences satisfy constant absolute risk aversion if and only if for all* $\boldsymbol{\pi} \in \mathcal{P}$, $RS(\boldsymbol{\pi}) + \beta \mathbf{1} \subseteq R$, $\beta \in \Re$. *Preference satisfy constant relative risk aversion if and only if or all* $\boldsymbol{\pi} \in \mathcal{P}$, $\mu RS(\boldsymbol{\pi}) \subseteq RS(\boldsymbol{\pi})$, $\mu > 0$. *Preferences satisfy linear tolerace if and only if* $RS(\boldsymbol{\pi}) = RS^0(\boldsymbol{\pi}) + RS^1(\boldsymbol{\pi})$ *with* $RS^0(\boldsymbol{\pi})$ *the reference set for* $B^0$ *and* $\mu RS^1(\boldsymbol{\pi}) \subseteq RS^1(\boldsymbol{\pi})$.

**Proof** *The proof is for constant absolute risk aversion, the remaining proofs are equally simple. By Theorem 11, if preferences satisfy constant absolute risk aversion,* $B(e, \mathbf{y}; \mathbf{1}) = B(0, \mathbf{y}; \mathbf{1}) - e$. *Hence,* $\hat{\mathbf{y}} \in \arg\inf_{\mathbf{y}} \{\boldsymbol{\pi}\mathbf{y} - B(0, \mathbf{y}; \mathbf{1})\} \Rightarrow \hat{\mathbf{y}} + e\mathbf{1} \in \arg\inf_{\mathbf{y}} \{\boldsymbol{\pi}\mathbf{y} - B(e, \mathbf{y}; \mathbf{1})\}$. *The converse follows by (2).*■

Karni (1985) generalizes the definition of the Pratt–Arrow risk premium for an outcome $\hat{\mathbf{y}} \in RS(\boldsymbol{\pi})$ to the "...maximum reduction in actuarial value that the decision maker is willing to accept to attain a point on the reference set rather than bear actuarially neutral risk". If $\boldsymbol{\pi}$ defines actuarially neutral risk, this can be computed dually as

$$v(\boldsymbol{\pi}, \hat{\mathbf{y}}, \mathbf{x}) = \boldsymbol{\pi}(\hat{\mathbf{y}} + \mathbf{x}) - E(\boldsymbol{\pi}, e(\hat{\mathbf{y}} + \mathbf{x})),$$

where $\boldsymbol{\pi}\mathbf{x} = 0$. Then by the assumption that $\hat{\mathbf{y}} \in RS(\boldsymbol{\pi})$,

$$v(\boldsymbol{\pi}, \hat{\mathbf{y}}, \mathbf{x}) = E(\boldsymbol{\pi}, e(\hat{\mathbf{y}})) - E(\boldsymbol{\pi}, e(\hat{\mathbf{y}} + \mathbf{x}))$$

for actually fair risks. Perhaps more familiarly, Karni's risk premium can be recognized as the Hicksian compensating variation of the actually fair risk, $\mathbf{x}$. Thus for arbitrarily small risks, we have the following dual approximation

$$v(\boldsymbol{\pi}, \hat{\mathbf{y}}, \mathbf{x}) \approx E_e(\boldsymbol{\pi}, e(\hat{\mathbf{y}}))\left[e(\hat{\mathbf{y}}) - e(\hat{\mathbf{y}} + \mathbf{x})\right].$$

This observation implies that one can always recast Karni's notions of more risk-averse and less risk-averse for general risk-averse preferences more neutrally by speaking in terms of a higher or lower willingness to pay for avoiding a gamble of the form $\boldsymbol{\pi}\mathbf{x} = 0$ around the reference set. And although we do not pursue the issue here, this observation implies that the vast literature on consumer surplus and other approximations to the compensating and equivalent variations can be applied to yield close approximations for $v(\boldsymbol{\pi}, \hat{\mathbf{y}}, \mathbf{x})$ (e.g., Diewert, 1992).

Parallel to $v(\boldsymbol{\pi}, \hat{\mathbf{y}}, \mathbf{x})$, one can define a generalized radial risk premium as

$$r(\boldsymbol{\pi}, \hat{\mathbf{y}}, \mathbf{x}) = \frac{E(\boldsymbol{\pi}, e(\hat{\mathbf{y}}))}{E(\boldsymbol{\pi}, e(\hat{\mathbf{y}} + \mathbf{x}))},$$

if one restricts attentions to preferences defined over $\Re^S_{++}$.

Our next result, which follows from Theorem 11, demonstrates that these generalized risk premiums assume particularly convenient forms in the presence of either constant absolute risk aversion or constant relative risk aversion.

**Corollary 20** *$v(\boldsymbol{\pi}, \hat{\mathbf{y}}, \mathbf{x}) = \hat{v}(\hat{\mathbf{y}}, \mathbf{x})$ for all $\hat{\mathbf{y}} \in RS(\boldsymbol{\pi})$, $\boldsymbol{\pi} \in \mathcal{P}$, and $\mathbf{x}$ such that $\sum_s \pi_s x_s = 0$ if and only if preferences satisfy constant absolute risk aversion. $r(\boldsymbol{\pi}, \hat{\mathbf{y}}, \mathbf{x}) = \hat{r}(\hat{\mathbf{y}}, \mathbf{x})$ for all $\hat{\mathbf{y}} \in RS(\boldsymbol{\pi}) \cap \Re^S_{++}$, $\boldsymbol{\pi} \in \mathcal{P}$, and $\mathbf{x}$ such that $\sum_s \pi_s x_s = 0$ if and only if preferences satisfy constant relative risk aversion.*

**Remark 3** *In the case of constant absolute risk aversion, $v(\boldsymbol{\pi}, \hat{\mathbf{y}}, \mathbf{x}) = e(\hat{\mathbf{y}}) - e(\hat{\mathbf{y}} + \mathbf{x})$, while in the case of constant relative risk aversion $r(\boldsymbol{\pi}, \hat{\mathbf{y}}, \mathbf{x}) = \frac{e(\hat{\mathbf{y}})}{e(\hat{\mathbf{y}}+\mathbf{x})}$.*

The Pratt-Arrow notion of the risk premium presumes that the reference set is the sure-thing vector. By the properties of the expected-value function and Lemma 4, we have the following relationship between $v(\boldsymbol{\pi}, \hat{\mathbf{y}}, \mathbf{x})$ and the standard Pratt-Arrow risk premium.

**Lemma 21** *$v(\boldsymbol{\pi}, \hat{\mathbf{y}}, \mathbf{x}) \geq \boldsymbol{\pi}(\hat{\mathbf{y}} + \mathbf{x}) - e(\hat{\mathbf{y}} + \mathbf{x})$, and the individual is risk-averse with respect to $\boldsymbol{\pi}$ if and only if $v(\boldsymbol{\pi}, \hat{\mathbf{y}}, \mathbf{x}) = \boldsymbol{\pi}(\hat{\mathbf{y}} + \mathbf{x}) - e(\hat{\mathbf{y}} + \mathbf{x})$.*

An important reason for defining a generalized risk premium is to permit the comparison of aversion to risk across individuals sharing a common reference set. Assuming individuals A and B have the same reference set for $\boldsymbol{\pi}$, individual A is said to be more risk-averse than B (in the sense of Karni (1985)) if

$$v^A(\boldsymbol{\pi}, \hat{\mathbf{y}}, \mathbf{x}) \geq v^B(\boldsymbol{\pi}, \hat{\mathbf{y}}, \mathbf{x}).$$

for all $\hat{\mathbf{y}} \in RS(\boldsymbol{\pi})$ and $\mathbf{x}$ such that $\sum_s \pi_s x_s = 0$.[6] Hence, for A to be more risk averse than B in the sense of Karni, A must be willing to forego more in actuarial value terms in order to avoid exposure

[6] It would be more appropriate to say locally more risk averse in the sense of Karni to reflect the fact that this only applies for a given ▯ .

to the actuarially fair risk, $\mathbf{x}$, than B. This leads us to the following dual characterization of 'more risk-averse' individuals. If A and B have the same reference set, A is more risk averse than B in the sense of Karni if and only if

$$E^A\left(\boldsymbol{\pi}, e^A\left(\hat{\mathbf{y}}+\mathbf{x}\right)\right) \leq E^B\left(\boldsymbol{\pi}, e^B\left(\hat{\mathbf{y}}+\mathbf{x}\right)\right)$$

for all $\hat{\mathbf{y}} \in RS\left(\boldsymbol{\pi}\right)$, and $\mathbf{x}$ such that $\sum_s \pi_s x_s = 0$. So, in other words, A is more risk averse than B if and only if A is willing to pay less than B for all actuarially fair departures from the reference set. When A and B have the same reference set, and A and B are both risk averse with respect to $\boldsymbol{\pi}$, so that $\{\alpha\mathbf{1} : \alpha \in \Re\} \subseteq RS\left(\boldsymbol{\pi}\right)$, Lemma 21 implies that Karni's notion of more risk averse corresponds locally to that contained in Theorem 6.

To permit a generalization of the Pratt-Arrow notions of decreasing, increasing, and constant absolute risk aversion for state-dependent preferences, Karni (1985) restricts attention to *autocomparable* preferences. Preferences are *autocomparable* if the reference set is affine.[7] To admit the possibility of multiple solutions to the expected-value problem, we say that an individual's preferences are autocomparable for $\boldsymbol{\pi}$ if $RS\left(\boldsymbol{\pi}\right) = RS^0\left(\boldsymbol{\pi}\right) + RS^1\left(\boldsymbol{\pi}\right)$ with $RS^0\left(\boldsymbol{\pi}\right) \subset Y^S$ and $RS^1\left(\boldsymbol{\pi}\right)$ a cone, i.e., $\mu RS^1\left(\boldsymbol{\pi}\right) \subseteq RS^1\left(\boldsymbol{\pi}\right), \mu > 0$. Preferences can be autocomparable for some $\boldsymbol{\pi}$ but not for others. For example, state-independent expected utility preferences are autocomparable for the subjective probabilities that parametrize the expected-utility function, but not necessarily for other probabilities. Hence, for general risk-averse preferences, it is of interest to isolate the classes of preferences which are autocomparable across all probability distributions. By Theorem 19, both constant absolute risk aversion and constant relative risk aversion ensure autocomparability for all $\boldsymbol{\pi} \in \mathcal{P}$. More generally,

**Corollary 22** *Preferences are autocomparable for all* $\boldsymbol{\pi} \in \mathcal{P}$ *if and only if they satisfy linear risk tolerance.*

For autocomparable preferences, decreasing absolute risk aversion in the sense of Karni (1985) requires

$$v\left(\boldsymbol{\pi}, \mathbf{y}\left(\boldsymbol{\pi}, e+\alpha\right), \mathbf{x}\right) \leq v\left(\boldsymbol{\pi}, \mathbf{y}\left(\boldsymbol{\pi}, e\right), \mathbf{x}\right)$$

for all $e$ and all $\alpha > 0$ and $\mathbf{x}$ such that $\sum_s \pi_s x_s = 0$.[8] Focusing on preferences autocomparable for all $\boldsymbol{\pi}$, we obtain:

[7] Karni (1985) uses the term linear, but his definition is equivalent to requiring the elements of a particular point in $RS\left(\square\right)$ to be affine translates of one another.

[8] Again, it may be more precise to say decreasing willingness to pay for the avoidance of the fair gamble.

**Corollary 23** *Preferences are autocomparable for all* $\boldsymbol{\pi} \in \mathcal{P}$ *and exhibit decreasing (increasing, constant) absolute risk aversion in the sense of Karni if and only if they satisfy linear risk tolerance and*

$$e\left(\mathbf{y}\left(\boldsymbol{\pi},e+\alpha\right)+\mathbf{x}\right)-e\left(\mathbf{y}\left(\boldsymbol{\pi},e\right)+\mathbf{x}\right)\geq\left(\leq,=\right)\alpha,$$

*for all* $e$ *and all* $\alpha > 0$ *and any given* $\boldsymbol{\pi} \in \mathcal{P}$, *and* $\mathbf{x}$ *such that* $\sum_s \pi_s x_s = 0$.

**Proof** By Corollary 22 and linear risk tolerance,

$$v\left(\boldsymbol{\pi},\mathbf{y}\left(\boldsymbol{\pi},e+\alpha\right),\mathbf{x}\right)-v\left(\boldsymbol{\pi},\mathbf{y}\left(\boldsymbol{\pi},e\right),\mathbf{x}\right)=\alpha E^{1}\left(\boldsymbol{\pi}\right)+E^{1}\left(\boldsymbol{\pi}\right)\left[e\left(\mathbf{y}\left(\boldsymbol{\pi},e\right)+\mathbf{x}\right)-e\left(\mathbf{y}\left(\boldsymbol{\pi},e+\alpha\right)+\mathbf{x}\right)\right].\blacksquare$$

We close this section with a result that relates measures of risk aversion based on willingness to pay to our dual notion of risk aversion.

**Theorem 24** $E^A\left(\boldsymbol{\pi}, e^A\left(\mathbf{y}\right)\right) \leq E^B\left(\boldsymbol{\pi}, e^B\left(\mathbf{y}\right)\right)$ *for all* $\mathbf{y} \in Y^S$ *and all* $\boldsymbol{\pi} \in \mathcal{P}$ *if and only if : (a) A is more risk averse than B; (b)* $B^A\left(e,\mathbf{y};\mathbf{1}\right) \leq B^B\left(e,\mathbf{y};\mathbf{1}\right)$ *for all* $\mathbf{y}$ *and* $e$; *(c)* $E^A\left(\boldsymbol{\pi},e\right) \geq E^B\left(\boldsymbol{\pi},e\right)$ *for all* $\pi$ *and* $e$.

**Proof** By duality

$$\begin{aligned}V\left(e^{B}\left(\mathbf{y}\right)\right) &= \cap_{\pi}\left\{\hat{\mathbf{y}}:\boldsymbol{\pi}\mathbf{y}\geq E^{B}\left(\boldsymbol{\pi},e^{B}\left(\mathbf{y}\right)\right)\right\}\\ &\subseteq \cap_{\pi}\left\{\hat{\mathbf{y}}:\boldsymbol{\pi}\mathbf{y}\geq E^{A}\left(\boldsymbol{\pi},e^{A}\left(\mathbf{y}\right)\right)\right\}\\ &= V\left(e^{A}\left(\mathbf{y}\right)\right)\end{aligned}$$

by $E^A\left(\boldsymbol{\pi}, e^A\left(\mathbf{y}\right)\right) \leq E^B\left(\boldsymbol{\pi}, e^B\left(\mathbf{y}\right)\right)$. Hence, $e^B\left(\mathbf{y}\right)\mathbf{1} \in V\left(e^A\left(\mathbf{y}\right)\right)$ implying $e^A\left(\mathbf{y}\right) \leq e^B\left(\mathbf{y}\right)$. Apply Theorem 6.■

Theorem 24 establishes that if two individuals share a common reference set, and A is more risk averse than B in our dual sense, then A is also more risk averse in the sense of Karni. The converse does not apply.

# 3 Comparative Statics and $E\left(\boldsymbol{\pi},e\right)$

$E\left(\boldsymbol{\pi},e\right)$ is a particularly convenient representation of preferences in cases where the risk-neutral probabilities are exogenous to the individual. Such a case obviously pertains when the individual is a small participant in a complete contingent claims markets, or when the conditions of the arbitrage pricing theorem apply. The risk-neutral probabilities can be treated as the normalized Arrow-Debreu contingent-claim prices.[9] In the case of complete contingent claims, the importance of linear risk tolerance in obtaining several standard results in finance theory including results on the existence of aggregate consumers and the two-fund separation theorem has already been recognized (Milne, 1995; DeTemple and Gottardi 1998). In this section, we show how standard results in consumer theory can be coupled with our results on dual measures of aversion to risk to obtain results for general preferences.

In what follows, for simplicity, we assume that $E(\boldsymbol{\pi}, e)$ is smoothly differentiable,[10] and we restrict attention to $\mathbf{y} \in \Re_+^S$. As a convenient reference point, we also consider the special case of expected utility, where the certainty equivalent satisfies

$$u(e(\mathbf{y})) = \sum_{s \in \Omega} \hat{\pi}_s u(y_s).$$

Here $\hat{\boldsymbol{\pi}}$ is the vector of subjective probabilities and $u$ is a smooth increasing concave function. For this case,

$$E(\boldsymbol{\pi}, e) = \hat{E}(\boldsymbol{\pi}, u(e)),$$

where $\hat{E}(\boldsymbol{\pi}, u)$ is the expected-value function for the expected-utility functional.

Consider the case of a pure-exchange economy where the individual's endowment of the contingent claims is given by the vector $\mathbf{y}^*$. Her equilibrium welfare level, $e^*(\boldsymbol{\pi}\mathbf{y}^*, \boldsymbol{\pi})$, is determined as the implicit solution to her budget constraint:[11]

$$E(\boldsymbol{\pi}, e) = \boldsymbol{\pi}\mathbf{y}^*. \tag{3}$$

Using (2), her corresponding equilibrium holding of the contingent claims is given by

$$\mathbf{y}(\boldsymbol{\pi}\mathbf{y}^*, \boldsymbol{\pi}) = \nabla_{\boldsymbol{\pi}} E(\boldsymbol{\pi}, e^*(\boldsymbol{\pi}\mathbf{y}^*, \boldsymbol{\pi})), \tag{4}$$

where $\nabla$ represents the usual gradient with respect to the subscripted vector.

Expressions (3) and (4) offer a platform from which to conduct a variety of comparative-static experiments. Suppose, for example, that one is interested in calculating the individual's response

[9] If obtained via the arbitrage pricing theorem, they need not be unique.

[10] As pointed out above, there are many instances (CARA and linear risk tolerance) where the expected-value function will be smooth even though the primal representation is not.

[11] In more familiar terms, $e^*(\square\, \mathbf{y}^*, \square)$ is the individual's indirect utility function.

to a change in her endowment vector by the small amount $\triangle\mathbf{y}^*$. The effect on her equilibrium certainty equivalent is

$$\frac{\boldsymbol{\pi}\triangle\mathbf{y}^*}{E_e\left(\boldsymbol{\pi},e^*\left(\boldsymbol{\pi}\mathbf{y}^*,\boldsymbol{\pi}\right)\right)},$$

while the associated change in her holding of the contingent claims is given by

$$\boldsymbol{\pi}\triangle\mathbf{y}^*\frac{\nabla_{\boldsymbol{\pi}e}E\left(\boldsymbol{\pi},e^*\left(\boldsymbol{\pi}\mathbf{y}^*,\boldsymbol{\pi}\right)\right)}{E_e\left(\boldsymbol{\pi},e^*\left(\boldsymbol{\pi}\mathbf{y}^*,\boldsymbol{\pi}\right)\right)}.$$

This change in the contingent claims corresponds to movements along Karni's (1986) reference set for the probabilities $\boldsymbol{\pi}$ and to movements along Nau's (2001) wealth expansion path.

If the market's evaluation of her wealth change, $\boldsymbol{\pi}\triangle\mathbf{y}^*$, is positive, several observations follow. $E_e\left(\boldsymbol{\pi},e^*\left(\boldsymbol{\pi}\mathbf{y}^*,\boldsymbol{\pi}\right)\right)$ is the reciprocal of the marginal utility of income, and for small enough changes in the certainty equivalent it can be interpreted as the compensating variation of the change in $e^*\left(\boldsymbol{\pi}\mathbf{y}^*,\boldsymbol{\pi}\right)$ induced by the wealth change. Hence, it is positive. (This also follows from the properties of the expected-value function.) For the expected-utility model,

$$E_e\left(\boldsymbol{\pi},e\right)=\hat{E}_u\left(\boldsymbol{\pi},u\left(e\right)\right)u'\left(e\right).$$

Our dual concept of decreasing (increasing) relative risk aversion is exactly analogous to the concept of decreasing (increasing) average cost familiar from the theory of the firm. Hence, by direct extension of standard results from that theory, decreasing relative risk aversion locally requires

$$E_e\left(\boldsymbol{\pi},e\right)\leq r\left(\boldsymbol{\pi},e\right)\leq 1.$$

On the other hand, if there is decreasing absolute risk aversion, it must be locally true that

$$E_e\left(\boldsymbol{\pi},e\right)\leq 1.$$

Hence, decreasing relative risk aversion implies decreasing absolute risk aversion, but not the contrary. From these observations we conclude the following for a positive wealth change: if preferences exhibit decreasing absolute risk aversion, the certainty equivalent increases by more than the associated increase in wealth; if preferences exhibit decreasing relative risk aversion, the percentage change in the certainty equivalent is no less than the percentage change in wealth; if preferences exhibit increasing relative risk aversion but decreasing absolute risk aversion, the certainty equivalent increases more than the change in wealth, but the percentage change in the certainty equivalent is less than the percentage change in wealth; and, if preferences exhibit increasing absolute risk aversion, they must also exhibit increasing relative risk aversion. Hence, if preferences exhibit increasing absolute risk aversion both the absolute and the percentage change in the certainty equivalent must be less than the corresponding wealth change.

For the expected-utility model, the dual notion of decreasing risk aversion has a particularly simple interpretation. In that case, $\hat{E}_u(\boldsymbol{\pi},u)$ is the reciprocal of the marginal utility of income. Hence, under expected utility, decreasing absolute risk aversion requires that the marginal utility of the certainty equivalent be less than the marginal utility of income to the individual. Decreasing relative risk aversion, on the other hand, requires that[12]

$$\frac{u(e)\,\hat{E}_u(\boldsymbol{\pi},u(e))}{\hat{E}(\boldsymbol{\pi},u(e))}\frac{u'(e)\,e}{u(e)} \leq 1,$$

so that the elasticity of utility at the certainty equivalent is less than $\left(\frac{u(e)\hat{E}_u(\boldsymbol{\pi},u(e))}{\hat{E}(\boldsymbol{\pi},u(e))}\right)^{-1}$. By standard results from producer theory, this latter expression is the elasticity of size of the expected utility functional at the solution to the expected-value minimization problem. We leave it to the reader to rephrase the general consequences of decreasing risk aversion for the certainty equivalent in these terms for the expected-utility model.

The induced change in the $s$th contingent claim is positive if and only if the $s$th element of the vector, $\nabla_{\boldsymbol{\pi}e}E(\boldsymbol{\pi},e^*(\boldsymbol{\pi}\mathbf{y}^*,\boldsymbol{\pi}))$, is positive. This vector, in turn, measures the change in the compensated demands induced by a change in real income, as measured by the certainty equivalent. Hence, on the basis of well-known results in consumer theory, we conclude that demand for the contingent claim rises as a result of the wealth change if and only if it is a normal good. A necessary and sufficient condition for the state claims to be normal is that $B$ be supermodular in $(\mathbf{y},e)$ (Milgrom and Shannon, 1994). For general quasi-concave preferences, violations of these conditions are well understood, and the existence of inferior goods is well recognized. Hence general quasi-concave preferences over stochastic incomes can exhibit instances where contingent claims can fall as a result of an increase in income.

Compare this general result with what emerges under expected utility. Under expected utility, it is well-known that preferences are risk-averse if and only if there is a diminishing marginal utility of income. It also implies, however, that all contingent claims are normal for all $\boldsymbol{\pi}$ (including $\boldsymbol{\pi} \neq \hat{\boldsymbol{\pi}}$). First-order conditions for an interior solution to the expected-value maximization problem are

$$\pi_s - \hat{E}_u(\boldsymbol{\pi},u(e))\,\hat{\pi}_s u'\left(\hat{E}_s(\boldsymbol{\pi},u(e))\right), \quad s \in \Omega.$$

[12]Here for the sake of simplicity, we have taken $u > 0$ .

Differentiating with respect to $e$ and rearranging gives

$$\frac{\nabla_{\boldsymbol{\pi}e} E\left(\boldsymbol{\pi},e^*\left(\boldsymbol{\pi}\mathbf{y}^*,\boldsymbol{\pi}\right)\right)}{E_e\left(\boldsymbol{\pi},e^*\left(\boldsymbol{\pi}\mathbf{y}^*,\boldsymbol{\pi}\right)\right)} = \frac{\nabla_{\boldsymbol{\pi}u}\hat{E}_u\left(\boldsymbol{\pi},u\left(e^*\right)\right)}{\hat{E}_u\left(\boldsymbol{\pi},u\left(e^*\right)\right)} = -\left(\frac{\hat{E}_{uu}\left(\boldsymbol{\pi},u\left(e^*\right)\right)}{\hat{E}_u\left(\boldsymbol{\pi},u\left(e^*\right)\right)^2}\right)\left[\frac{u'\left(y_s\right)}{u''\left(y_s\right)}\right].$$

Because the expected-utility function is strictly concave in $\mathbf{y}$, its conjugate expected-value function is convex in $u$. Hence, it follows immediately that the induced change in the $sth$ contingent claim is of the same sign as the wealth change and just proportional to the individual's risk tolerance at that contingent claim. Put another way, under expected-utility all contingent claims are normal goods, and the relative adjustments in contingent claims are determined by their relative risk tolerances.

If preferences exhibit linear risk tolerance, the change in the holding of her contingent claims is given by

$$\frac{\boldsymbol{\pi}\triangle\mathbf{y}^*}{E^1\left(\boldsymbol{\pi}\right)}\nabla_{\boldsymbol{\pi}}E^1\left(\boldsymbol{\pi}\right),$$

which is positive in all its components. Hence, when the individual preferences are characterized by linear risk tolerance, all contingent claim demands rise. In the special case where preferences exhibit constant absolute risk aversion, all contingent claims holdings rise by the same small amount $\boldsymbol{\pi}\triangle\mathbf{y}^*$. This is the analogue of the well-known result that a wealth change does not change an individual with CARA preference purchases of the risky asset in the portfolio problem. More generally, in the case where preferences are translation homothetic but do not exhibit CARA, the relative impacts of an income change on contingent claims, measured by $\frac{u_i}{u_j}$ for all $i$ and $j$, are independent of real income and relative prices.

The welfare effect of a price change, $\triangle\boldsymbol{\pi}$, on the individual can be computed as

$$\frac{\left(\mathbf{y}^* - \nabla_{\boldsymbol{\pi}}E\left(\boldsymbol{\pi},e\right)\right)}{E_e\left(\boldsymbol{\pi},e\right)}\triangle\boldsymbol{\pi},$$

and its sign hinges upon whether the price change turns the terms-of-trade in favor of the individual or against the individual. Similarly, it also straightforward to show that the response of the individual's holdings of the contingent claims to a price change can be decomposed into a compensated demand effect and a real-income effect. The compensated demand effect is given by the Hessian matrix of $E$ in the probabilities. It follows immediately that, in the case of linear risk tolerance, and its special cases of CARA and CRRA, each demand responds negatively to a change in its own price.

In this framework it is easy to assess how an individual's welfare is affected by an increase or decrease in the riskiness of the contingent claims prices. For example, suppose that $\boldsymbol{\pi}^o \in \cap_e\{\partial B\left(e, e\mathbf{1};\mathbf{1}\right)\}$, and that for those subjective probabilities the contingent claims prices undergo the simple mean preserving change

$$d\pi_1 = -\frac{\pi_2^o}{\pi_1^o} d\pi_2.$$

If $\pi_2 > \pi_1$ and $d\pi_2$, this corresponds to a simple mean preserving spread. Welfare, therefore, only rises if[13]

$$\frac{(y_2 - E_2)}{\pi_2^o} - \frac{(y_1 - E_1)}{\pi_1^o} \geq 0.$$

## 4 Primal characterization of asset demands

The problem of primal characterizations of the demand for risky assets is of considerable interest. Moreover, since all economic decisions under uncertainty share with portfolio choices the property that they may regarded as choices of state-contingent income or consumption vectors, general results on asset demands may be extended to apply to a wide range of problems such as consumption–savings choices (Sandmo 1970), output decisions for owner-operated firms (Sandmo 1971), and labor supply decisions (Block and Heineke 1973).

Various measures of willingness to pay for risky assets will be useful in what follows. Following Luenberger (1996), we define the compensating and equivalent benefits (in units of $\mathbf{g}$) by

$$\begin{aligned} CB(\mathbf{y}^0, \mathbf{y}^1) &= B(e^0, \mathbf{y}^1; \mathbf{g}) - B\left(e^0, \mathbf{y}^0; \mathbf{g}\right), \\ EB(\mathbf{y}^0, \mathbf{y}^1) &= B(e^1, \mathbf{y}^1; \mathbf{g}) - B(e^1, \mathbf{y}^0; \mathbf{g}). \end{aligned}$$

The compensating and equivalent benefits can be recognized as generalizations of Allais's measure of disposable surplus (Luenberger, 1996; Chambers, 2001). In words, they are, respectively, the units of the reference risky asset, $\mathbf{g}$, that can be subtracted from $\mathbf{y}^1$ to leave the individual just indifferent to $\mathbf{y}^0$, and the units of the reference bundle that must be added to $\mathbf{y}^0$ to make him indifferent to $\mathbf{y}^1$. When preferences are strictly increasing in all state-contingent incomes, these measures reduce to

$$\begin{aligned} CB(\mathbf{y}^0, \mathbf{y}^1) &= B(e^0, \mathbf{y}^1; \mathbf{g}) \\ EB(\mathbf{y}^0, \mathbf{y}^1) &= -B(e^1, \mathbf{y}^0; \mathbf{g}) \end{aligned}$$

[13] A sufficient condition for this inequality to always be satisfied is that

$$E(\square, e) - \square\, \mathbf{y}^*,$$

which is referred to as the trade-expenditure function in the international trade literature, be generalized Schur concave for $\square^o$ in the sense of Chambers and Quiggin (1997).

because $B(e^i, \mathbf{y}^i; \mathbf{g}) = 0$. Chambers and Färe (1998) establish:

**Lemma 25** *$CB = EB$ globally if and only if preferences are translation homothetic (in the direction of $\mathbf{g}$).*

Special cases of the compensating benefit and equivalent benefit which have received attention in the literature on preferences over stochastic outcomes (Nau, 2001) are the *buying price* of the asset $\mathbf{z}$ (at $\mathbf{y}$)

$$P_b(\mathbf{z}, \mathbf{y}) = B(e(\mathbf{y}), \mathbf{y} + \mathbf{z}, \mathbf{1}),$$

and the *selling price* of the asset

$$P_s(\mathbf{z}, \mathbf{y}) = -B(e(\mathbf{y} + \mathbf{z}), \mathbf{y}, \mathbf{1}).$$

In words, the buying price and the selling price of the asset are the number of units of the traditionally safe asset (that is, the one identified by the $45^o$ line) that the producer is willing to give or accept for the asset.[14] The selling price corresponds to the 'option-price' of $\mathbf{z}$ familiar from the literature on environmental economics (Henry, 1974; Graham, 1981).

By the properties of the translation function, the buying price of an asset is positive if and only if the selling price of the asset is positive, and if and only if $e(\mathbf{y} + \mathbf{z}) \geq e(\mathbf{y})$. By the definition of the superdifferential

$$\boldsymbol{\pi}\mathbf{z} \geq P_b(\mathbf{z}, \mathbf{y}) \quad \forall \boldsymbol{\pi} \in \boldsymbol{\pi}(\mathbf{y})$$

and

$$\boldsymbol{\pi}\mathbf{z} \geq P_s(\mathbf{z}, \mathbf{y}) \quad \forall \boldsymbol{\pi} \in \boldsymbol{\pi}(\mathbf{y} + \mathbf{z}).$$

Because the buying price and selling price of the asset are expressed in units of the certainty vector, it is not surprising that their relationship to one another is closely related to other measures of risk which are normalized in the same fashion. However, because of the presence of real-wealth effects, the buying and the selling price generally diverge from one another, although, as Nau (2001) shows, they are approximately equal for small enough asset vectors. More precisely, we have as a direct consequence of Lemma 25 and results in Quiggin and Chambers (1998) that the buying price and the selling price of the asset always equal one another only when these real wealth effects are completely neutral. More formally:[15]

[14]It is straightforward to define a buying price and a selling price denominated in terms of a risky asset rather than the traditionally safe asset. However, since such measures do not appear to have been recognized in the literature, we leave their consideration to a future paper.

[15]For the sake of completeness, we provide a direct proof.

**Theorem 26** *The buying price and the selling price of an asset are always equal if and only if preferences are characterized by constant absolute risk aversion.*

**Proof** *If preferences satisfy constant absolute risk aversion, by Theorem 11*

$$B(e, \mathbf{y}; \mathbf{1}) = B(0, \mathbf{y}; \mathbf{1}) - e$$

*whence*

$$\begin{aligned} P_b(\mathbf{z}, \mathbf{y}) &= B(0, \mathbf{y} + \mathbf{z}, \mathbf{1}) - e(\mathbf{y}) \\ &= e(\mathbf{y} + \mathbf{z}) - B(0, \mathbf{y}, \mathbf{1}) \\ &= -B(e(\mathbf{y} + \mathbf{z}), \mathbf{y}; \mathbf{1}) \\ &= P_s(\mathbf{z}, \mathbf{y}). \end{aligned}$$

*Conversely, if for all* $\mathbf{y}$ *and* $\mathbf{z}$

$$-B(e(\mathbf{y} + \mathbf{z}), \mathbf{y}; \mathbf{1}) = B(e(\mathbf{y}), \mathbf{y} + \mathbf{z}; \mathbf{1})$$

*the result may be derived by setting* $\mathbf{y} = \mathbf{0}$ *to obtain*

$$-B(e(\mathbf{z}), \mathbf{0}; \mathbf{1}) = B(0; \ \mathbf{z}; \mathbf{1}) . \blacksquare$$

Nau (2001) defines the *marginal price of a financial asset* in the smooth case as the two-sided directional derivative of the preference structure, $\boldsymbol{\pi}(\mathbf{y})\mathbf{z}$ (in the smooth case, $\boldsymbol{\pi}(\mathbf{y})$is a singleton). Because we allow for nondifferentiable preferences, we thus define two marginal prices of a financial asset using the more general notion of one-sided directional derivatives. We have

$$\begin{aligned} P_m^+(\mathbf{z}, \mathbf{y}) &= B'(e(\mathbf{y}), \mathbf{y}, \mathbf{1}; \mathbf{z}) \\ P_m^-(\mathbf{z}, \mathbf{y}) &= -B'(e(\mathbf{y}), \mathbf{y}, \mathbf{1}; -\mathbf{z}). \end{aligned}$$

We have

$$\begin{aligned} P_m^+(\mathbf{z}, \mathbf{y}) &= \inf_{\boldsymbol{\pi} \in \boldsymbol{\pi}(\mathbf{y})} \{\boldsymbol{\pi}\mathbf{z}\} \\ P_m^-(\mathbf{z}, \mathbf{y}) &= \sup_{\boldsymbol{\pi} \in \boldsymbol{\pi}(\mathbf{y})} \{\boldsymbol{\pi}\mathbf{z}\}, \end{aligned}$$

whence

$$P_m^-(\mathbf{z}, \mathbf{y}) \geq P_m^+(\mathbf{z}, \mathbf{y}) \geq P_b(\mathbf{z}, \mathbf{y}). \tag{5}$$

Expression (5) generalizes Nau's Proposition 1 to the nondifferentiable case. As Nau notes, the fact that the marginal price of the asset always exceeds the selling price of the asset is a straightforward consequence of the convexity of the individual's at-least-as-good sets reflected here by the concavity of the translation function and the properties of the superdifferential.

Following Nau (2001), an asset $\mathbf{z}$ is *neutral if* $P_m^+(\mathbf{z},\mathbf{y}) = 0$. By (5), (1), and Lemma 1, an asset is neutral if and only if its buying and selling prices are negative, from which one concludes that $e(\mathbf{y}) \geq e(\mathbf{y}+\mathbf{z})$. More generally, we can completely characterize the marginal price of the asset in both the nondifferentiable and differentiable cases by exploiting the basic properties of one-sided directional derivatives for concave functions.

**Lemma 27** *$P_m^+(\mathbf{z},\mathbf{y})$ is a nondecreasing, concave, and positively linearly homogeneous function of $\mathbf{z}$. $P_m^-(\mathbf{z},\mathbf{y})$ is a nondecreasing, convex, and positively linearly homogeneous function of $\mathbf{z}$. If preferences are differentiable at $\mathbf{y}$,*

$$P_m(\mathbf{z},\mathbf{y}) = P_m^-(\mathbf{z},\mathbf{y}) = P_m^+(\mathbf{z},\mathbf{y})$$

*is a nondecreasing, linear function of $\mathbf{z}$.*

Constant absolute risk aversion, the case where the buying and the selling price of an asset are always equal, is particularly interesting not only because of its centrality to much of the literature on primal measures of risk aversion, but also because it allows us to glean some further insight into the nature of the connection between state-claim prices and the risk-neutral probabilities. We have the following extension of a result originally due in the consumer context to Chambers (2001):

**Theorem 28** *If the buying and selling price of an asset are always equal, and $B$ is generalized quadratic then*

$$P_b(\mathbf{z},\mathbf{y}) = P_s(\mathbf{z},\mathbf{y}) = \frac{1}{2}\left(\boldsymbol{\pi}(\mathbf{y}) + \boldsymbol{\pi}(\mathbf{y}+\mathbf{z})\right)\mathbf{z}.$$

**Proof** *By Theorem 26, if the buying and selling price of the asset are always equal then $B(e,\mathbf{y};\mathbf{1}) = B(0,\mathbf{y};\mathbf{1}) - e$. Because $B(e(\mathbf{y}),\mathbf{y};\mathbf{1}) = 0$, in this case*

$$\begin{aligned} P_b(\mathbf{z},\mathbf{y}) &= B(e(\mathbf{y}),\mathbf{y}+\mathbf{z};\mathbf{1}) - B(e(\mathbf{y}),\mathbf{y};\mathbf{1}) \\ &= B(0,\mathbf{y}+\mathbf{z};\mathbf{1}) - B(0,\mathbf{y};\mathbf{1}). \end{aligned}$$

*Diewert's (1976b) quadratic lemma applied here then yields*

$$B(0,\mathbf{y}+\mathbf{z};\mathbf{1}) - B(0,\mathbf{y};\mathbf{1}) = \frac{1}{2}\left(\nabla_{\mathbf{y}}B(0,\mathbf{y}+\mathbf{z};\mathbf{1}) + \nabla_{\mathbf{y}}B(0,\mathbf{y};\mathbf{1})\right)\mathbf{z},$$

*which gives the result.*■

Perhaps the easiest way to interpret Theorem 28 is to recall the interpretation of the risk-neutral probabilities as the individual's internal state-claim prices. Theorem 28 shows that in the case where wealth effects are neutral (CARA preferences make them neutral everywhere), the individual's internal price of the asset is approximately equal to Hicks' (1945-46) many-market consumer surplus measure for the asset in terms of those state-claims prices. Moreover, in the presence of complete state-claims markets (or nearly complete state-claims markets), the individual's internal price of the asset for small enough changes will be well approximated by the market value of the asset. In a complete state-claims markets, all individuals will equate their relative internal state-claim prices to the market's, and hence the market's evaluation can be used to evaluate the value of the asset.

Moreover, because the quadratic provides a second-order flexible approximation to any smooth preference structure, an immediate consequence of Theorem 28 is that $\frac{1}{2}(\boldsymbol{\pi}(\mathbf{y})+\boldsymbol{\pi}(\mathbf{y}+\mathbf{z}))\mathbf{z}$ represents a superlative indicator, in the sense of Diewert (1976a), for the buying price of the asset under CARA preferences. Theorem 28 provides exact results for the buying price of the asset under specific restrictions on preferences. If these conditions do not hold, a more standard approach can be taken to approximating the buying price of the asset using second-order Taylor's series approximations for the case of differentiable preferences. Because $B(e(\mathbf{y}), \mathbf{y}; \mathbf{1}) = 0$,

$$\begin{aligned} P_b(\mathbf{z}, \mathbf{y}) &= B(e(\mathbf{y}), \mathbf{y}+\mathbf{z}; \mathbf{1}) - B(e(\mathbf{y}), \mathbf{y}; \mathbf{1}) \\ &\approx \nabla_{\mathbf{y}} B(e(\mathbf{y}), \mathbf{y}; \mathbf{1})\mathbf{z} + \frac{1}{2}\mathbf{z}'\nabla_{\mathbf{yy}} B(e(\mathbf{y}), \mathbf{y}; \mathbf{1})\mathbf{z} \\ &= \pi(\mathbf{y})\mathbf{z} + \frac{1}{2}\mathbf{z}'\nabla_{\mathbf{y}} p(e(\mathbf{y}), \mathbf{y})\mathbf{z} \\ &= P_m(\mathbf{z}, \mathbf{y}) + \frac{1}{2}\mathbf{z}'\nabla_{\mathbf{y}} p(e(\mathbf{y}), \mathbf{y})\mathbf{z}. \end{aligned}$$

Similarly, presuming the existence of a set of subjective probabilities and letting $\mu = \boldsymbol{\pi}(\mathbf{1})\mathbf{y}$, one can use similar methods to arrive at the standard approximation for the risk premium in terms of the translation function.

When preferences exhibit either constant absolute risk aversion, constant relative risk aversion, or constant risk aversion, we can use Theorems 11 and 14 to strengthen these results.

**Corollary 29** *If preferences exhibit constant absolute risk aversion,*

$$\begin{aligned} P_b(\mathbf{z}+\beta\mathbf{1}, \mathbf{y}) &= P_b(\mathbf{z}, \mathbf{y}) + \beta, \\ P_b(\mathbf{z}, \mathbf{y}+\beta\mathbf{1}) &= P_b(\mathbf{z}, \mathbf{y}), \\ P_m^+(\mathbf{z}, \mathbf{y}+\beta\mathbf{1}) &= P_m^+(\mathbf{z}, \mathbf{y}), \\ P_m^-(\mathbf{z}, \mathbf{y}+\beta\mathbf{1}) &= P_m^-(\mathbf{z}, \mathbf{y}), \quad \beta \in \Re. \end{aligned}$$

*If preferences exhibit constant relative risk aversion,*

$$\begin{aligned} P_b(\mu\mathbf{z},\mu\mathbf{y}) &= P_b(\mathbf{z},\mathbf{y}), \\ P_s(\mu\mathbf{z},\mu\mathbf{y}) &= P_s(\mathbf{z},\mathbf{y}), \\ P_m^+(\mathbf{z},\mu\mathbf{y}) &= P_m^+(\mathbf{z},\mathbf{y}), \\ P_m^-(\mathbf{z},\mu\mathbf{y}) &= P_m^-(\mathbf{z},\mathbf{y}), \quad \mu > 0. \end{aligned}$$

*If preferences exhibit constant risk aversion, then either*

$$P_b(\mathbf{z},\mathbf{y}) = \min\{y_1+z_1,...,y_S+z_S\} - \min\{y_1,y_2,...,y_S\},$$

$$\begin{aligned} P_m^+(\mathbf{z},\mathbf{y}) &= \inf_{s\in\Omega^*}\{\mathbf{e}_s\mathbf{z}\}, \\ P_m^-(\mathbf{z},\mathbf{y}) &= \sup_{s\in\Omega^*}\{\mathbf{e}_s\mathbf{z}\}, \end{aligned}$$

*or*

$$P_b(\mathbf{z},\mathbf{y}) = \inf\left\{\boldsymbol{\pi}(\mathbf{y}+\mathbf{z}) : \boldsymbol{\pi} \in \mathcal{P}\right\} - \inf\left\{\boldsymbol{\pi}\mathbf{y} : \boldsymbol{\pi} \in \mathcal{P}\right\}$$

$$\begin{aligned} P_m^+(\mathbf{z},\mathbf{y}) &= \inf\{\boldsymbol{\pi}(\mathbf{y})\mathbf{z}\}, \\ P_m^-(\mathbf{z},\mathbf{y}) &= \sup\{\boldsymbol{\pi}(\mathbf{y})\mathbf{z}\}, \end{aligned}$$

*where*

$$\boldsymbol{\pi}(\mathbf{y}) = \arg\inf_{\mathcal{P}}\left\{\boldsymbol{\pi}\mathbf{y} - \inf\left\{\boldsymbol{\pi}\mathbf{y} : \boldsymbol{\pi} \in \mathcal{P}\right\}\right\}.$$

**Proof** *The first two parts are trivial. By theorem 14, if preferences exhibit constant risk aversion then either*

$$e(\mathbf{y}) = \min\{y_1,y_2,...,y_S\},$$

*or*

$$e(\mathbf{y}) = \inf\left\{\boldsymbol{\pi}\mathbf{y} : \boldsymbol{\pi} \in \mathcal{P}\right\}.$$

*Consider first the case of maximin preferences. Then*

$$\begin{aligned} P_b(\mathbf{z},\mathbf{y}) &= B(0,\mathbf{y}+\mathbf{z};\mathbf{1}) - B(0,\mathbf{y};\mathbf{1}) \\ &= \min\{y_1+z_1,...,y_S+z_S\} - \min\{y_1,y_2,...,y_S\}, \end{aligned}$$

*and*

$$\boldsymbol{\pi}(\mathbf{y}) = \mathcal{P}$$

*if* $\mathbf{y} = e\mathbf{1}$.

*If* $\mathbf{y} \neq e\mathbf{1}$,

$$\boldsymbol{\pi}(\mathbf{y}) = conv\{\mathbf{e}_s : s \in \Omega^*\}.$$

*Thus, for either* $\mathbf{y} = e\mathbf{1}$ *or* $\mathbf{y} \neq e\mathbf{1}$

$$\begin{aligned} P_m^+(\mathbf{z}, \mathbf{y}) &= \inf_{s \in \Omega^*} \{\mathbf{e}_s \mathbf{z}\}, \\ P_m^-(\mathbf{z}, \mathbf{y}) &= \sup_{s \in \Omega^*} \{\mathbf{e}_s \mathbf{z}\}. \end{aligned}$$

*For* $e(\mathbf{y}) = \inf\{\boldsymbol{\pi}\mathbf{y} : \boldsymbol{\pi} \in \mathcal{P}\}$, *the result follows by conjugacy.*■

Nau (2001) has defined, for smooth preferences, a local measure of risk aversion, which generalizes the Pratt-Arrow risk premium, but which is not based upon either the traditional riskless asset or the subjective probabilities. Hence, it applies either in the presence of undiversifiable background risk or for state-dependent preferences. In his terms, the buying risk premium is the difference between the marginal price of the asset and the buying price. To account for the possibility of nondifferentiable preferences, we thus have two relevant notions of a buying risk premium

$$\begin{aligned} r_b^-(\mathbf{z}, \mathbf{y}) &= P_m^-(\mathbf{z}, \mathbf{y}) - P_b(\mathbf{z}, \mathbf{y}) \geq 0 \\ r_b^+(\mathbf{z}, \mathbf{y}) &= P_m^+(\mathbf{z}, \mathbf{y}) - P_b(\mathbf{z}, \mathbf{y}) \geq 0, \end{aligned}$$

with $r_b^-(\mathbf{z}, \mathbf{y}) \geq r_b^+(\mathbf{z}, \mathbf{y}) \geq 0$ as a consequence of (5).

The properties of one-sided directional derivatives and the translation function immediately lead to a number of results on these notions of the buying risk premium that extend the results of Nau (2001) on smooth preferences. We have:

**Theorem 30** $r_b^-(\mathbf{z}, \mathbf{y})$ *is convex in* $\mathbf{z}$. *If preferences are differentiable at* $\mathbf{y}$,

$$r_b^-(\mathbf{z}, \mathbf{y}) = r_b^+(\mathbf{z}, \mathbf{y}) = r_b(\mathbf{z}, \mathbf{y}) \quad \forall \mathbf{z},$$

*and* $r_b(\mathbf{z}, \mathbf{y})$ *is convex in* $\mathbf{z}$, *and*

$$r_b(\mathbf{z}, \mathbf{y}) \approx -\frac{1}{2}\mathbf{z}' \nabla_{\mathbf{y}} p(e(\mathbf{y}), \mathbf{y}) \mathbf{z}.$$

**Theorem 31** *If preferences satisfy constant absolute risk aversion*

$$\begin{aligned} r_b^+(\mathbf{z}, \mathbf{y} + \beta\mathbf{1}) &= r_b^+(\mathbf{z}, \mathbf{y}), \quad \beta \in \Re, \\ r_b^-(\mathbf{z}, \mathbf{y} + \beta\mathbf{1}) &= r_b^-(\mathbf{z}, \mathbf{y}), \quad \beta \in \Re. \end{aligned}$$

***Corollary 32*** *If preferences satisfy constant absolute risk aversion and $B$ is generalized quadratic then*

$$r_b^+(\mathbf{z},\mathbf{y}) = r_b^-(\mathbf{z},\mathbf{y}) = \frac{1}{2}(\boldsymbol{\pi}(\mathbf{y}) - \boldsymbol{\pi}(\mathbf{y}+\mathbf{z}))\,\mathbf{z}.$$

**Theorem 33** *$A$ is more risk-averse than $B$ if and only if for any $\mathbf{z}, e$,*

$$\begin{aligned} P_b^A(\mathbf{z}, e\mathbf{1}) &\leq P_b^B(\mathbf{z}, e\mathbf{1}) \\ P_s^A(-\mathbf{z}, \mathbf{z}+e\mathbf{1}) &\geq P_b^B(-\mathbf{z}, \mathbf{z}+e\mathbf{1}) \end{aligned}$$

## 5 Concluding comments

Yaari (1969) and Peleg and Yaari (1975) observed that an individual's preferences over uncertain outcomes can be represented in terms of supporting hyperplanes to convex indifference surfaces. This paper investigates general, ordinal risk-averse preference functionals over uncertain outcomes in terms of a convenient cardinal equivalent, the benefit (translation) function, and its concave conjugate, the expected-value function. 'Risk-neutral probabilities' are deduced from and characterized by the translation function. Dual notions of risk aversion are defined in terms of the 'risk-neutral probabilities' and the expected-value function. These notions are shown to be equivalent to existing primal notions, and their structural consequences for the translation function, the expected-value function, and the risk-neutral probabilities are characterized. Constant absolute risk aversion, constant relative risk aversion, constant risk aversion, and linear risk tolerance are all characterized for general, risk-averse preferences and shown to correspond to various notions of homotheticity. MMEV preferences are shown to exhaustively characterize the class of quasi-concave constant risk averse preferences, and all constant risk averse preference functionals exhibit the plunging property of Yaari's (1987) dual model. Karni's (1985) notions of a reference asset and risk premium are conveniently characterized in dual terms. The use of the expected-value function in comparative-static analysis for the general portfolio problem is illustrated. Nau's (2001) buying price, selling price, and marginal prices, and the generalized risk premium are generalized and characterized for the class of Yaari (1969) risk-averse ordinal preferences. Constant absolute risk aversion is shown to be necessary and sufficient for the buying and selling price to be everywhere equal, and in that case we develop an exact and superlative index of the value of a risky asset in the presence of background risk that extends the standard Pratt–Arrow approximations. An appropriate version of the generalized risk premium of the risky asset is shown to be convex in the risky asset.

In our discussion of the buying price, the selling price, and the marginal price of the asset, all evaluations have been done in terms of units of the traditionally safe asset. This was done to maximize comparability with existing results in the literature. However, as already suggested, it is an easy matter to generalize these results by using a different reference asset (a different direction for the benefit function) on which to base the respective price calculations. The main substantive change required in the analysis is a renormalization of the supporting state-claim prices that would allow them to be interpretable as risk-neutral probabilities. This ability should prove particularly important in instances where the presence of undiversifiable background risk or state-dependent preferences make the traditionally safe asset a poor choice for price or welfare calculations. Even more generally, however, it is also apparent that complete dual and primal theories of risk aversion can be developed in the presence of undiversifiable background risk or state-dependent preferences in terms of an arbitrary reference asset. We leave that development to later work.

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